

Gallai-Edmonds Structure Theorem for Weighted Matching Polynomial

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Abstract

In this paper, we prove the Gallai-Edmonds structure theorem for the most general matching polynomial. Our result implies the Parter-Wiener theorem and its recent generalization about the existence of principal submatrices of a Hermitian matrix whose graph is a tree.

KEYWORDS: matching polynomial, characteristic polynomial, Gallai-Edmond decomposition, Hermitian matrices, Parter-Wiener theorem

1 Introduction

Recently, Chen and Ku [3] proved an analogue of the celebrated Gallai-Edmonds structure theorem for general roots of the matching polynomial. Their result implies that every connected vertex transitive graph has simple matching polynomial roots. Subsequently, following a line of investigation pursued by Lovász and Plummer [22], Ku and Wong wrote a series of papers [14, 15, 16, 17, 18, 19] to develop a matching theory for general roots of the matching polynomial. In this paper, we shall prove the Gallai-Edmonds structure theorem for the most general matching polynomial. Surprisingly, our result implies the Parter-Wiener theorem and its recent generalization by Johnson, Duarte and Saiago [11] about the existence of principal submatrices of a Hermitian matrix whose graph is a tree.

All graphs in this paper are simple and finite. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$ respectively. Recall that an r -*matching* in a graph G is a set of r edges, no two of which have a vertex in common. The number of r -matchings in G will be denoted by $p(G, r)$. We set $p(G, 0) = 1$ and define the *matching polynomial* of G by

$$\mu(G, x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r p(G, r) x^{n-2r},$$

where $n = |V(G)|$.

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In this paper we shall consider weighted versions of the matching polynomial. From now on, we assign a non-zero complex number $w(e)$ to every edge e of our graph G (we shall give a reason why we do not want w to take zero value later). We can view w as a function on $E(G)$ and call w the *edge weight function*. We also denote an edge by e_{uv} to emphasize that the edge has endpoints u and v . For each complex number $y = a + bi \in \mathbb{C}$, we denote its conjugate by $\bar{y} = a - bi$ and its magnitude by $|y| = \sqrt{a^2 + b^2}$. Also for any set S , we denote the number of elements in S by $|S|$. Although the notations for the magnitude of a complex number and the number of element in a set look similar, they will not cause any confusion.

For each $A \subseteq E(G)$, we define $w(A) = \prod_{e \in A} w(e)$. We set $w(\emptyset) = 1$. Let $\mathcal{M}(G)$ denote the set of all matchings of G including the empty set \emptyset . The *edge weighted matching polynomial* of G is defined by

$$\mu_w(G, x) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} |w(M)|^2 x^{n-2|M|}.$$

We denote the set of all r -matchings in G by $M_r(G)$ and set $M_0(G) = \{\emptyset\}$. The following lemma is obvious from the definition.

Lemma 1.1.

$$\mu_w(G, x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \left(\sum_{M \in M_r(G)} |w(M)|^2 \right) x^{n-2r},$$

where $n = |V(G)|$. □

Using Lemma 1.1, it is not hard to deduce the followings.

Lemma 1.2. *For any edge weight function w , zero is a root of $\mu_w(G, x)$ if and only if G does not have a perfect matching.* □

Lemma 1.3. *Suppose $w(e) = 1$ for all $e \in E(G)$. Then $\mu_w(G, x) = \mu(G, x)$.* □

By Lemma 1.2, the fact that zero is a root of $\mu_w(G, x)$ depends only on the structure of the graph G and does not depend on the edge weight function. By Lemma 1.3, if the edge weight function takes only the value 1, then the edge weighted matching polynomial is the usual matching polynomial.

Let $u \in V(G)$. The graph obtained from G by deleting the vertex u and all edges that contain u will be denoted by $G \setminus u$. The weight function on $G \setminus u$ is induced by the weight function w on G . Inductively if $u_1, \dots, u_k \in V(G)$, $G \setminus u_1 \dots u_k = (G \setminus u_1 \dots u_{k-1}) \setminus u_k$. For convenience if H is a subgraph of G then we shall denote $G \setminus V(H)$ by $G \setminus H$. If $e_1, \dots, e_m \in E(G)$ then the graph obtained from G by deleting all the edges e_1, \dots, e_m will be denoted by $G - e_1 \dots e_m$. The weight function on $G - e_1 \dots e_m$ is induced by the weight function w on G .

If we were to allow w to take zero value then $\mu_w(G, x) = \mu_w(G - e_1 \dots e_m, x)$ where $w(e_1) = \dots = w(e_k) = 0$. So we may remove the edges with zero weight and the resulting graph has the same edge weighted matching polynomial. This is the reason why we do not allow w to take zero value.

The edge weighted matching polynomial $\mu_w(G, x)$ is a special case of the original multivariate matching polynomial introduced by Heilmann and Lieb [9], who proved that all roots of $\mu_w(G, x)$ are real ([9, Theorem 4.2]). As a consequence, the roots of the usual matching polynomial $\mu(G, x)$ are real (Lemma 1.3). This fact was also proved by Godsil in his book [5, Corollary 1.2 on p. 97]

via the classical recursive approach (see also [8, Corollary 5.2]). Recently, by generalizing Foata's combinatorial proof of the Mehler formula for Hermite polynomials to matching polynomials, Lass [21, Corollary on p. 439] proved that all the roots of $\mu_w(G, x)$ are real.

Now let us further generalize the edge weighted matching polynomial by assigning a real number $w_1(u)$ to every vertex u of our graph G (we allow w_1 to take zero value). We can view w_1 as a function on $V(G)$ and call w_1 the *vertex weight function*. The pair (w, w_1) will be called the *weight function*. For each non-empty set $S \subseteq V(G)$, let $H_G(S)$ be the subgraph of G induced by the vertices in S , that is $V(H_G(S)) = S$ and $e_{uv} \in E(H_G(S))$ if and only if $e_{uv} \in E(G)$ and $u, v \in S$.

For each $S \subseteq V(G)$, we define $w_1(G \setminus S) = \prod_{u \in V(G) \setminus S} w_1(u)$. We set $w_1(\emptyset) = 1$, $H_G(\emptyset) = \emptyset$ and $\mu_w(\emptyset, x) = 1$. The *weighted matching polynomial* of G is defined by

$$\eta_{(w, w_1)}(G, x) = \sum_{S \subseteq V(G)} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x).$$

It turns out that the weighted matching polynomial can be rewritten as

$$\eta_{(w, w_1)}(G, x) = \sum_{M \in \mathcal{M}(G)} \left(\left(\prod_{e \in M} w(e) \right) \left(\prod_{u \in V(G) \setminus V(M)} (x - w_1(u)) \right) \right)$$

which was proved by Averbouch and Makowsky [1] to be the most general nontrivial polynomial satisfying the matching polynomial recurrence relations.

Example 1.4. Let G be the graph in Figure 1. Let $w(e_{u_1 u_2}) = 2 + i$, $w_1(u_1) = 1$ and $w_1(u_2) = 3$. Note that all the possible subsets of $V(G)$ are $S_1 = \emptyset$, $S_2 = \{u_1\}$, $S_3 = \{u_2\}$ and $S_4 = \{u_1, u_2\}$. Now $\mu_w(H_G(S_1), x) = 1$, $\mu_w(H_G(S_2), x) = x$, $\mu_w(H_G(S_3), x) = x$ and $\mu_w(H_G(S_4), x) = x^2 - |2+i|^2 = x^2 - 5$. Also $w_1(G \setminus S_1) = w_1(u_1)w_1(u_2) = 3$, $w_1(G \setminus S_2) = w_1(u_2) = 3$, $w_1(G \setminus S_3) = w_1(u_1) = 1$ and $w_1(G \setminus S_4) = 1$. Therefore $\eta_{(w, w_1)}(G, x) = (x^2 - 5) - (1)x - (3)x + 3 = x^2 - 4x - 2$.

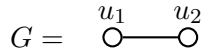


Figure 1.

□

Example 1.5. Let G be the graph in Figure 2. Let $w(e_{v_1 v_2}) = 1 + 2i$, $w(e_{v_2 v_3}) = 2 - 7i$, $w(e_{v_1 v_3}) = -3 + 2i$, $w_1(v_1) = 1$, $w_1(v_2) = 2$ and $w_1(v_3) = 3$. Note that all the possible subsets of $V(G)$ are $S_1 = \emptyset$, $S_2 = \{v_1\}$, $S_3 = \{v_2\}$, $S_4 = \{v_3\}$, $S_5 = \{v_1, v_2\}$, $S_6 = \{v_1, v_3\}$, $S_7 = \{v_2, v_3\}$, $S_8 = \{v_1, v_2, v_3\}$. Now $\mu_w(H_G(S_1), x) = 1$, $\mu_w(H_G(S_2), x) = x$, $\mu_w(H_G(S_3), x) = x$, $\mu_w(H_G(S_4), x) = x$, $\mu_w(H_G(S_5), x) = x^2 - 5$, $\mu_w(H_G(S_6), x) = x^2 - 13$, $\mu_w(H_G(S_7), x) = x^2 - 53$ and $\mu_w(H_G(S_8), x) = x^3 - (5 + 13 + 53)x = x^3 - 71x$. Also $w_1(G \setminus S_1) = w_1(v_1)w_1(v_2)w_1(v_3) = 6$, $w_1(G \setminus S_2) = w_1(v_2)w_1(v_3) = 6$, $w_1(G \setminus S_3) = w_1(v_1)w_1(v_3) = 3$, $w_1(G \setminus S_4) = w_1(v_1)w_1(v_2) = 2$, $w_1(G \setminus S_5) = w_1(v_3) = 3$, $w_1(G \setminus S_6) = w_1(v_2) = 2$, $w_1(G \setminus S_7) = w_1(v_1) = 1$ and $w_1(G \setminus S_8) = 1$. Therefore $\eta_{(w, w_1)}(G, x) = (x^3 - 71x) - (x^2 - 53) - 2(x^2 - 13) - 3(x^2 - 5) + 2(x) + 3(x) + 6(x) - 6 = x^3 - 6x^2 - 60x + 88$.

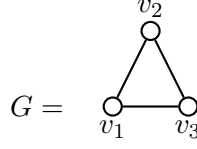


Figure 2.

□

For consistency, we set $\eta_{(w,w_1)}(\emptyset, x) = 1$. The following three lemmas are obvious.

Lemma 1.6. *If $w_1(u) = 0$ for all $u \in V(G)$ then*

$$\eta_{(w,w_1)}(G, x) = \mu_w(G, x).$$

□

Lemma 1.7. *Let $u_1, \dots, u_m \in V(G)$ be such that $w_1(u_1) = \dots = w_1(u_m) = 0$. Then*

$$\eta_{(w,w_1)}(G, x) = \sum_{\substack{S \subseteq V(G), \\ \{u_1, \dots, u_m\} \subseteq S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x).$$

□

Lemma 1.8. *The degree of $\eta_{(w,w_1)}(G, x)$ is equal to the degree of $\mu_w(G, x)$, which is $|V(G)|$.*

□

Let G_1 and G_2 be graphs with weight function (w, w_1) and (w', w'_1) , respectively. The two graphs are said to be *weight-isomorphic* if there is a bijection $f : V(G_1) \rightarrow V(G_2)$ such that

- (a) $e_{f(u)f(v)} \in E(G_2)$ if and only if $e_{uv} \in E(G_1)$,
- (b) $w'(e_{f(u)f(v)}) = w(e_{uv})$ for all $e_{uv} \in E(G_1)$,
- (c) $w'_1(f(u)) = w_1(u)$ for all $u \in V(G_1)$.

Note that if conditions (b) and (c) are removed then this is just the ‘usual’ isomorphism.

Example 1.9. Let G_1 and G_2 be the graphs in Figure 3. The edge weight functions for both graphs take value 1 for all the edges, whereas the vertex weight functions are as stated. Note that they are not weight-isomorphic (even though they are isomorphic in the ‘usual’ sense when the weights are removed).

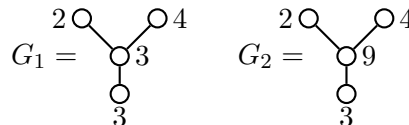


Figure 3.

□

The following lemma can be proved easily.

Lemma 1.10. *Let G_1 and G_2 be graphs with weight function (w, w_1) and (w', w'_1) , respectively. If G_1 is weight-isomorphic to G_2 , then $\eta_{(w, w_1)}(G_1, x) = \eta_{(w', w'_1)}(G_2, x)$.* □

Now by Lemma 1.6, the weighted matching polynomial $\eta_{(w, w_1)}(G, x)$ is a generalization of the edge weighted matching polynomial $\mu_w(G, x)$. So it is quite natural to ask whether the roots of $\eta_{(w, w_1)}(G, x)$ are real or not. In Section 3, we give an affirmative answer using Godsil's approach [5] (Corollary 3.3). This generalizes the result of Lass [21, Corollary on p. 439].

Let G be a graph with $V(G) = \{1, 2, \dots, n\}$ and $B_{(w, w_1)}(G) = [b_{uv}]$ be the $n \times n$ matrix with

$$b_{uv} = \begin{cases} w(e_{uv}), & \text{if } e_{uv} \in E(G) \text{ and } u < v; \\ w_1(u), & \text{if } u = v; \\ \overline{w(e_{vu})}, & \text{if } e_{vu} \in E(G) \text{ and } u > v; \\ 0, & \text{otherwise.} \end{cases}$$

We call $B_{(w, w_1)}(G)$ the *weighted adjacency matrix* of G . Note that $B_{(w, w_1)}(G)$ is a *Hermitian* matrix, that is $\overline{b_{uv}} = b_{vu}$ for all u, v . The *weighted characteristic polynomial* of G is defined by

$$\phi_{(w, w_1)}(G, x) = \det(xI - B_{(w, w_1)}(G)).$$

Example 1.11. Let G and (w, w_1) be as given in Example 1.4. Here we assume $V(G) = \{1, 2\}$ where $u_1 \equiv 1$ and $u_2 \equiv 2$. Then

$$B_{(w, w_1)}(G) = \begin{pmatrix} 1 & 2+i \\ 2-i & 3 \end{pmatrix},$$

and $\phi_{(w, w_1)}(G, x) = x^2 - 4x - 2$. □

Example 1.12. Let G and (w, w_1) be as given in Example 1.5. Here we assume $V(G) = \{1, 2, 3\}$ where $v_1 \equiv 1$, $v_2 \equiv 2$ and $v_3 \equiv 3$. Then

$$B_{(w, w_1)}(G) = \begin{pmatrix} 1 & 1+2i & -3+2i \\ 1-2i & 2 & 2-7i \\ -3-2i & 2+7i & 3 \end{pmatrix},$$

and $\phi_{(w, w_1)}(G, x) = x^3 - 6x^2 - 60x + 196$. □

Note that if $w(e) = 1$ for all $e \in E(G)$ and $w_1(u) = 0$ for all $u \in V(G)$, we recover the usual characteristic polynomial of G and $B_{(w, w_1)}(G)$ is the usual adjacency matrix. Godsil and Gutman [8, Theorem 4] first proved the relation between the characteristic polynomial of G and its matching polynomial. In Section 2, we shall show that similar relation holds for weighted characteristic polynomial and weighted matching polynomial (Theorem 2.10). As a consequence, the weighted characteristic polynomial of a graph and its the weighted matching polynomial are identical if and only if the graph is a forest, provided that the edge weight function w is positive real-valued (Corollary 2.14).

We would like to remark that ‘ordering’ in $V(G)$ is very important. Different ordering in $V(G)$ could give different weighted characteristic polynomial (see Example 1.13). This also means that

in general weight-isomorphic graphs might not have the same weighted characteristic polynomials. However if G is a tree or G is any graph with real valued edge weight function, then the ‘ordering’ in $V(G)$ will have no effect on its weighted characteristic polynomial (Corollary 2.15 and Corollary 2.16, respectively).

Example 1.13. Let G_1 and G_2 be the graphs in Figure 4. Let $V(G_1) = \{u_1, u_2, u_3, u_4\}$ and $V(G_2) = \{v_1, v_2, v_3, v_4\}$. Suppose the vertex weight functions for both graphs take value 0 for all the vertices, whereas the edge weight functions are as given in the figure.

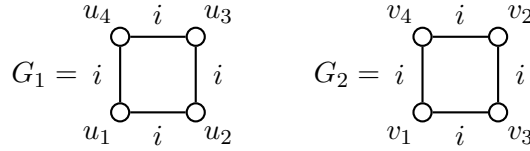


Figure 4.

Now order the vertices of G_1 as follows: $u_1 \equiv 1, u_2 \equiv 2, u_3 \equiv 3, u_4 \equiv 4$. Then

$$B_{(w,w_1)}(G_1) = \begin{pmatrix} 0 & i & 0 & i \\ -i & 0 & i & 0 \\ 0 & -i & 0 & i \\ -i & 0 & -i & 0 \end{pmatrix},$$

and $\phi_{(w,w_1)}(G_1, x) = x^4 - 4x^2 + 4$.

Suppose we order the vertices of G_2 as follows: $v_1 \equiv 1, v_2 \equiv 2, v_3 \equiv 3, v_4 \equiv 4$. Then

$$B_{(w,w_1)}(G_2) = \begin{pmatrix} 0 & 0 & i & i \\ 0 & 0 & i & i \\ -i & -i & 0 & 0 \\ -i & -i & 0 & 0 \end{pmatrix},$$

and $\phi_{(w,w_1)}(G_2, x) = x^4 - 4x^2$. So even though G_1 is weight-isomorphic to G_2 , $\phi_{(w,w_1)}(G_1, x) \neq \phi_{(w,w_1)}(G_2, x)$. \square

We shall denote the multiplicity of θ as a root of $\eta_{(w,w_1)}(G, x)$ and $\mu_w(G, x)$ by $\text{mult}(\theta, G, \eta_{(w,w_1)})$ and $\text{mult}(\theta, G, \mu_w)$ respectively. In Section 4, we classify the vertices of G with respect to θ using Godsil’s approach [7, Section 3] and study their properties. In Section 5, we develop a Gallai-Edmonds decomposition associated to a root θ of the weighted matching polynomial (Corollary 5.12 and Corollary 5.13). In Section 6, we discuss the connection of our result with the classical Gallai-Edmonds decomposition which is associated to root $\theta = 0$. In Section 7, we deduce the Parter-Weiner theorem and its generalization.

2 Weighted matching polynomial and weighted characteristic polynomial

It is not difficult to verify the following recurrence relations of $\mu_w(G, x)$ following the proof in [5, Theorem 1.1 on p. 2]. The sketch of the proofs are provided.

Lemma 2.1. *Recurrence for $\mu_w(G, x)$. ($v \sim u$ means u is adjacent to v)*

- (a) $\mu_w(G \cup H, x) = \mu_w(G, x)\mu_w(H, x)$ where G and H are disjoint graphs.
- (b) $\mu_w(G, x) = \mu_w(G - e_{uv}, x) - |w(e_{uv})|^2 \mu_w(G \setminus uv, x)$ if e_{uv} is an edge of G .
- (c) $\mu_w(G, x) = x \mu_w(G \setminus u, x) - \sum_{v \sim u} |w(e_{uv})|^2 \mu_w(G \setminus uv, x)$.
- (d) $\frac{d}{dx}(\mu_w(G, x)) = \sum_{v \in V(G)} \mu_w(G \setminus v, x)$.

Proof. (a) Note that every r -matching in $G \cup H$ consists of an s -matching in G and an $r - s$ -matching in H . So for each $M \in M_r(G \cup H)$, $M = M_1 \cup M_2$ for some $M_1 \in M_s(G)$ and $M_2 \in M_{r-s}(H)$. Part (a) follows from Lemma 1.1, by noticing that

$$\begin{aligned} \sum_{M \in M_r(G \cup H)} |w(M)|^2 &= \sum_{s=0}^r \sum_{\substack{M_1 \in M_s(G), \\ M_2 \in M_{r-s}(H)}} |w(M_1 \cup M_2)|^2 \\ &= \sum_{s=0}^r \left(\sum_{M_1 \in M_s(G)} |w(M_1)|^2 \right) \left(\sum_{M_2 \in M_{r-s}(H)} |w(M_2)|^2 \right). \end{aligned}$$

(b) Let $P_r(e_{uv}) = \{M \in M_r(G) : e_{uv} \in M\}$. Note that if $M \in P_r(e_{uv})$, then $M \setminus \{e_{uv}\}$ is an $(r - 1)$ -matching in $G \setminus uv$, i.e. $M \setminus \{e_{uv}\} \in M_{r-1}(G \setminus uv)$. Also $M_r(G) \setminus P_r(e_{uv}) = M_r(G - e_{uv})$. Thus $M_r(G) = P_r(e_{uv}) \cup M_r(G - e_{uv})$. Part (b) follows from Lemma 1.1 by noticing that

$$\begin{aligned} \sum_{M \in M_r(G)} |w(M)|^2 &= \sum_{M \in M_r(G - e_{uv})} |w(M)|^2 + \sum_{M \in P_r(e_{uv})} |w(M)|^2 \\ &= \sum_{M \in M_r(G - e_{uv})} |w(M)|^2 + |w(e_{uv})|^2 \sum_{M \in M_{r-1}(G \setminus uv)} |w(M)|^2. \end{aligned}$$

(c) Note that $M_r(G) = M_r(G \setminus u) \cup (\bigcup_{v \sim u} P_r(e_{uv}))$, where $P_r(e_{uv}) = \{M \in M_r(G) : e_{uv} \in M\}$. So part (c) follows from Lemma 1.1 by noticing that

$$\begin{aligned} \sum_{M \in M_r(G)} |w(M)|^2 &= \sum_{M \in M_r(G \setminus u)} |w(M)|^2 + \sum_{v \sim u} \sum_{M \in P_r(e_{uv})} |w(M)|^2 \\ &= \sum_{M \in M_r(G \setminus u)} |w(M)|^2 + \sum_{v \sim u} |w(e_{uv})|^2 \sum_{M \in M_{r-1}(G \setminus uv)} |w(M)|^2. \end{aligned}$$

(d) Let $|V(G)| = n$. Then

$$\frac{d}{dx}(\mu_w(G, x)) = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r (n - 2r) \left(\sum_{M \in M_r(G)} |w(M)|^2 \right) x^{n-1-2r}.$$

Let

$$T_r(G) = \{(M, v) \in M_r(G) \times V(G) : v \text{ is not contained in any of the edges in } M\}.$$

Let us calculate the sum $\sum_{(M,v) \in T_r(G)} |w(M)|^2$ in two ways. First we fix M and count the number of v . Since M contains exactly r edges and each of the edges contains exactly 2 vertices, the number of vertices that are not contained in any of the edges in M is equal to $n - 2r$. Therefore $\sum_{(M,v) \in T_r(G)} |w(M)|^2 = (n - 2r) \left(\sum_{M \in M_r(G)} |w(M)|^2 \right)$.

Second we fix v and count the number of M . This is the number of r -matching in $G \setminus v$. Therefore $\sum_{(M,v) \in T_r(G)} |w(M)|^2 = \sum_{v \in V(G)} \sum_{M \in M_r(G \setminus v)} |w(M)|^2$. Part (d) then follows from Lemma 1.1. \square

Theorem 2.2. *Recurrence for $\eta_{(w,w_1)}(G, x)$. ($v \sim u$ means u is adjacent to v)*

(a) $\eta_{(w,w_1)}(G_1 \cup G_2, x) = \eta_{(w,w_1)}(G_1, x) \eta_{(w,w_1)}(G_2, x)$ where G_1 and G_2 are disjoint graphs.

(b) $\eta_{(w,w_1)}(G, x) = \eta_{(w,w_1)}(G - e_{uv}, x) - |w(e_{uv})|^2 \eta_{(w,w_1)}(G \setminus uv, x)$ if e_{uv} is an edge of G .

(c) $\eta_{(w,w_1)}(G, x) = (x - w_1(u)) \eta_{(w,w_1)}(G \setminus u, x) - \sum_{v \sim u} |w(e_{uv})|^2 \eta_{(w,w_1)}(G \setminus uv, x)$.

(d) $\frac{d}{dx}(\eta_{(w,w_1)}(G, x)) = \sum_{v \in V(G)} \eta_{(w,w_1)}(G \setminus v, x)$.

Proof. (a) For each $S \subseteq V(G_1 \cup G_2)$, $S = S_1 \cup S_2$ with $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$. Also by part (a) of Lemma 2.1, $\mu_w(H_{G_1 \cup G_2}(S), x) = \prod_{j=1}^2 \mu_w(H_{G_j}(S_j), x)$. Therefore

$$\begin{aligned} & (-1)^{|V((G_1 \cup G_2) \setminus S)|} w_1((G_1 \cup G_2) \setminus S) \mu_w(H_{G_1 \cup G_2}(S), x) \\ &= \prod_{j=1}^2 (-1)^{|V(G_j \setminus S_j)|} w_1(G_j \setminus S_j) \mu_w(H_{G_j}(S_j), x), \end{aligned}$$

and

$$\begin{aligned} \eta_{(w,w_1)}(G_1 \cup G_2, x) &= \sum_{S_1 \subseteq V(G_1)} \sum_{S_2 \subseteq V(G_2)} \prod_{j=1}^2 (-1)^{|V(G_j \setminus S_j)|} w_1(G_j \setminus S_j) \mu_w(H_{G_j}(S_j), x) \\ &= \eta_{(w,w_1)}(G_1, x) \eta_{(w,w_1)}(G_2, x). \end{aligned}$$

(b) Note that if $S \subseteq V(G)$ then either $\{u, v\} \subseteq S$ or $\{u, v\} \not\subseteq S$. Therefore

$$\begin{aligned} \eta_{(w,w_1)}(G, x) &= \sum_{\substack{S \subseteq V(G), \\ \{u,v\} \subseteq S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x) + \\ &\quad \sum_{\substack{S \subseteq V(G), \\ \{u,v\} \not\subseteq S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x). \end{aligned}$$

Now if $\{u, v\} \subseteq S$ then $e_{uv} \in E(H_G(S))$. So by part (b) of Lemma 2.1,

$$\begin{aligned} \sum_{\substack{S \subseteq V(G), \\ \{u, v\} \subseteq S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x) = \\ \left(\sum_{\substack{S \subseteq V(G), \\ \{u, v\} \subseteq S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S) - e_{uv}, x) \right) - \\ \left(|w(e_{uv})|^2 \sum_{\substack{S \subseteq V(G), \\ \{u, v\} \subseteq S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S) \setminus uv, x) \right). \end{aligned}$$

On the other hand, by setting $G' = G - e_{uv}$ we have

$$\begin{aligned} \eta_{(w, w_1)}(G', x) = \sum_{\substack{S \subseteq V(G'), \\ \{u, v\} \subseteq S}} (-1)^{|V(G' \setminus S)|} w_1(G' \setminus S) \mu_w(H_{G'}(S), x) + \\ \sum_{\substack{S \subseteq V(G'), \\ \{u, v\} \not\subseteq S}} (-1)^{|V(G' \setminus S)|} w_1(G' \setminus S) \mu_w(H_{G'}(S), x). \end{aligned}$$

Note that $(-1)^{|V(G' \setminus S)|} w_1(G' \setminus S) = (-1)^{|V(G \setminus S)|} w_1(G \setminus S)$. Furthermore if $\{u, v\} \not\subseteq S$, then $H_{G'}(S) = H_G(S)$ and $\mu_w(H_{G'}(S), x) = \mu_w(H_G(S), x)$. Therefore

$$\begin{aligned} \sum_{\substack{S \subseteq V(G'), \\ \{u, v\} \not\subseteq S}} (-1)^{|V(G' \setminus S)|} w_1(G' \setminus S) \mu_w(H_{G'}(S), x) = \\ \sum_{\substack{S \subseteq V(G), \\ \{u, v\} \not\subseteq S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x). \end{aligned}$$

Also if $\{u, v\} \subseteq S$ then $H_{G'}(S) = H_G(S) - e_{uv}$. Therefore

$$\begin{aligned} \eta_{(w, w_1)}(G, x) = \eta_{(w, w_1)}(G - e_{uv}, x) - \\ \left(|w(e_{uv})|^2 \sum_{\substack{S \subseteq V(G), \\ \{u, v\} \subseteq S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S) \setminus uv, x) \right). \end{aligned}$$

Now for each $S \subseteq V(G)$ and $\{u, v\} \subseteq S$, $S = S_1 \cup \{u, v\}$ where $S_1 \subseteq V(G \setminus uv)$. Note also that $H_G(S) \setminus uv = H_{G \setminus uv}(S_1)$ and $G \setminus S = (G \setminus uv) \setminus S_1$. Hence $\eta_{(w, w_1)}(G, x) = \eta_{(w, w_1)}(G - e_{uv}, x) - |w(e_{uv})|^2 \eta_{(w, w_1)}(G \setminus uv, x)$.

(c) Let v_1, \dots, v_k be all the vertices adjacent to u in G and $g(S) = (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x)$. For a set $T \subseteq \{v_1, \dots, v_k\}$, let $N(T) = \{S \subseteq V(G) : u \in S \text{ and if } v \sim u \text{ in } H_G(S) \text{ then } v \in T\}$. Then

$$\eta_{(w, w_1)}(G, x) = \sum_{\substack{S \subseteq V(G), \\ u \notin S}} g(S) + \sum_{T \subseteq \{v_1, \dots, v_k\}} \sum_{S \in N(T)} g(S).$$

For each $S \subseteq V(G)$ and $u \notin S$, we have $S \subseteq V(G \setminus u)$, and vice versa. Therefore $(-1)^{|V(G \setminus S)|} = -(-1)^{|V((G \setminus u) \setminus S)|}$, $w_1(G \setminus S) = w_1(u)w_1((G \setminus u) \setminus S)$ and $H_G(S) = H_{G \setminus u}(S)$. So

$$\sum_{\substack{S \subseteq V(G), \\ u \notin S}} g(S) = -w_1(u)\eta_{(w, w_1)}(G \setminus u, x).$$

Let $T \subseteq \{v_1, \dots, v_k\}$ (note that T can be empty set). By part (c) of Lemma 2.1, for each $S \subseteq V(G)$ such that $S \in N(T)$, we have

$$g(S) = x(-1)^{|V(G \setminus S)|}w_1(G \setminus S)\mu_w(H_G(S) \setminus u, x) - \sum_{v \in T} |w(e_{uv})|^2(-1)^{|V(G \setminus S)|}w_1(G \setminus S)\mu_w(H_G(S) \setminus uv, x).$$

Furthermore $S = \{u\} \cup S_1$ for some $S_1 \subseteq G \setminus u$ and $T \subseteq S_1$. Also $(-1)^{|V(G \setminus S)|} = (-1)^{|V((G \setminus u) \setminus S_1)|}$, $w_1(G \setminus S) = w_1((G \setminus u) \setminus S_1)$ and $H_G(S) \setminus u = H_{G \setminus u}(S_1)$. When S_1 runs through all the subsets of $V(G \setminus u)$, T runs through all the subsets of $\{v_1, \dots, v_k\}$. Therefore

$$\begin{aligned} & x \left(\sum_{T \subseteq \{v_1, \dots, v_k\}} \sum_{S \in N(T)} (-1)^{|V(G \setminus S)|}w_1(G \setminus S)\mu_w(H_G(S) \setminus u, x) \right) \\ &= x \left(\sum_{S_1 \subseteq V(G \setminus u)} (-1)^{|V((G \setminus u) \setminus S_1)|}w_1((G \setminus u) \setminus S_1)\mu_w(H_{(G \setminus u)}(S_1), x) \right) \\ &= x \left(\eta_{(w, w_1)}(G \setminus u, x) \right). \end{aligned}$$

Also

$$\begin{aligned} & \left(\sum_{T \subseteq \{v_1, \dots, v_k\}} \sum_{S \in N(T)} \sum_{v \in T} |w(e_{uv})|^2(-1)^{|V(G \setminus S)|}w_1(G \setminus S)\mu_w(H_G(S) \setminus uv, x) \right) \\ &= \sum_{v \sim u} |w(e_{uv})|^2 \left(\sum_{S_2 \subseteq V(G \setminus uv)} (-1)^{|V((G \setminus uv) \setminus S_2)|}w_1((G \setminus uv) \setminus S_2)\mu_w(H_{(G \setminus uv)}(S_2), x) \right) \\ &= \sum_{v \sim u} |w(e_{uv})|^2 \eta_{(w, w_1)}(G \setminus uv, x), \end{aligned}$$

where the first equality holds by comparing each term on the left and right sides of the equations: if $T = \emptyset$ then $\sum_{v \in T} |w(e_{uv})|^2(-1)^{|V(G \setminus S)|}w_1(G \setminus S)\mu_w(H_G(S) \setminus uv, x) = 0$. So we may assume $T \neq \emptyset$. For a fixed $v \in T$, the term $|w(e_{uv})|^2(-1)^{|V(G \setminus S)|}w_1(G \setminus S)\mu_w(H_G(S) \setminus uv, x)$ is on the left side of the equation. Note that $S = S_2 \cup \{u, v\}$ with $S_2 \subseteq V(G \setminus uv)$. Also $(-1)^{|V(G \setminus S)|} = (-1)^{|V((G \setminus uv) \setminus S_2)|}$, $w_1(G \setminus S) = w_1((G \setminus uv) \setminus S_2)$ and $H_G(S) \setminus uv = H_{G \setminus uv}(S_2)$. Therefore

$$\begin{aligned} |w(e_{uv})|^2(-1)^{|V(G \setminus S)|}w_1(G \setminus S)\mu_w(H_G(S) \setminus uv, x) &= \\ |w(e_{uv})|^2(-1)^{|V((G \setminus uv) \setminus S_2)|}w_1((G \setminus uv) \setminus S_2)\mu_w(H_{(G \setminus uv)}(S_2), x), \end{aligned}$$

which is a term on the right side of the equation. It is not hard to see that the terms on the left side is in one-to-one correspondence with the terms on the right.

Hence we have $\eta_{(w,w_1)}(G, x) = (x - w_1(u))\eta_{(w,w_1)}(G \setminus u, x) - \sum_{v \sim u} |w(e_{uv})|^2 \eta_{(w,w_1)}(G \setminus uv, x)$.

(d) Note that

$$\begin{aligned} \frac{d}{dx}(\eta_{(w,w_1)}(G, x)) &= \sum_{S \subseteq V(G)} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \frac{d}{dx} \mu_w(H_G(S), x) \\ &= \sum_{S \subseteq V(G)} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \sum_{v \in S} \mu_w(H_G(S) \setminus v, x), \end{aligned}$$

where the second equality follows from part (d) of Lemma 2.1. Note that

$$\begin{aligned} \sum_{S \subseteq V(G)} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \sum_{v \in S} \mu_w(H_G(S) \setminus v, x) \\ &= \sum_{v \in V(G)} \left(\sum_{S_1 \subseteq V(G \setminus v)} (-1)^{|V((G \setminus v) \setminus S_1)|} w_1((G \setminus v) \setminus S_1) \mu_w(H_{G \setminus v}(S_1), x) \right) \\ &= \sum_{v \in V(G)} \eta_{(w,w_1)}(G \setminus v, x), \end{aligned}$$

where the first equality holds by comparing each term on the left and right sides of the equations: for a fixed S and $v \in S$, the term $(-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S) \setminus v, x)$ is on the left side of the equation. Note that $S = S_1 \cup \{v\}$ with $S_1 \subseteq V(G \setminus v)$. Also $(-1)^{|V(G \setminus S)|} = (-1)^{|V((G \setminus v) \setminus S_1)|}$, $w_1(G \setminus S) = w_1((G \setminus v) \setminus S_1)$ and $H_G(S) \setminus v = H_{G \setminus v}(S_1)$. Therefore

$$\begin{aligned} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S) \setminus v, x) &= \\ &= (-1)^{|V((G \setminus v) \setminus S_1)|} w_1((G \setminus v) \setminus S_1) \mu_w(H_{G \setminus v}(S_1), x), \end{aligned}$$

which is a term on the right side of the equation. It is not hard to see that the terms on the left side is in one-to-one correspondence with the terms on the right. Hence the proof is completed. \square

Definition 2.3. An *elementary* graph is a disjoint union of single edges (K_2) or cycles (C_r).

A *spanning elementary subgraph* of a graph is an elementary subgraph that contains all the vertices of the graph.

We denote $\text{comp}(G)$ as the number of components in G .

For convenience, we shall write $w(H) = \prod_{e \in E(H)} w(e)$ for any subgraph H of G .

Let $V(G) = \{1, 2, \dots, n\}$. Let $v_1 v_2 \dots v_m v_1$, $m \geq 3$ be a cycle C in G . We set

$$w_2(C) = b_{v_1 v_2} b_{v_2 v_3} \dots b_{v_{m-1} v_m} b_{v_m v_1} + b_{v_1 v_m} b_{v_m v_{m-1}} \dots b_{v_3 v_2} b_{v_2 v_1},$$

where b_{uv} is the uv -entry in the weighted adjacency matrix $B_{(w,w_1)}(G)$. Note that $w_2(C) = b + \bar{b}$ where $b = b_{v_1 v_2} b_{v_2 v_3} \dots b_{v_{m-1} v_m} b_{v_m v_1}$. So $w_2(C)$ is a real number. The following lemma is obvious.

Lemma 2.4. *If the edge weight function w is positive real-valued, then $w_2(C) > 0$ for any cycle C in G .* \square

Now let us extend w_2 to the union of disjoint cycles. Let C_1, C_2, \dots, C_k be disjoint cycles in G and $C = C_1 \cup C_2 \cup \dots \cup C_k$. We set $w_2(C) = \prod_{j=1}^k w_2(C_j)$. We are ready to prove the next lemma whose non-weighted version was first observed by Harary [2, Proposition 7.2].

Lemma 2.5. Suppose $w_1(u) = 0$ for all $u \in V(G)$. Let Γ be the set of all spanning elementary subgraphs of G and $|V(G)| = n$. Then

$$\det B_{(w,w_1)}(G) = (-1)^n \sum_{\gamma \in \Gamma} (-1)^{\text{comp}(\gamma)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma),$$

where C_γ is the union of all the cycles in γ . In particular $\det B_{(w,w_1)}(G)$ is a real number.

Proof. Let $B_{(w,w_1)}(G) = [b_{uv}]$. Recall that $\det B_{(w,w_1)}(G) = \sum_{\pi \in S_n} \text{sign}(\pi) \prod_{u=1}^n b_{u\pi(u)}$ (see [23, Definition 1.2.2 on p. 6]) where S_n is the set of all permutations on $V(G) = \{1, 2, \dots, n\}$. Note that $b_{uu} = w_1(u) = 0$. So the term $\prod_{u=1}^n b_{u\pi(u)}$ vanishes if $b_{u\pi(u)} = 0$ for some u , that is either $\pi(u) = u$, or $\pi(u) \neq u$ and $e_{u\pi(u)}$ is not an edge in G . Therefore each non-vanishing term corresponds to a disjoint union of edges and cycles, which is a spanning subgraph of G . Furthermore the π that corresponds to the non-vanishing term can be written as a product of disjoint cycles of length at least 2 which is actually in correspondence to a spanning elementary subgraph of G (the fact that every $\pi \in S_n$ can be written as a product of disjoint cycles can be found in [4, Exercise 1.2.5 on p. 3]).

Let $S \subseteq S_n$ be the set of all π for which $\prod_{u=1}^n b_{u\pi(u)} \neq 0$. Let $f : S \rightarrow \Gamma$ be defined by $f(\pi) = \gamma$ where γ is the spanning elementary subgraph corresponds to π . Let $\gamma \in \Gamma$. First let us find $\prod_{u=1}^n b_{u\pi(u)}$ for each $\pi \in f^{-1}(\gamma)$. Let $\pi \in f^{-1}(\gamma)$. Let $u_1 u_2$ be an edge (K_2) in γ . Then in the decomposition of π , it must have the cycle $(u_1 \ u_2)$. Let $v_1 v_2 v_3 \dots v_{m-1} v_m v_1$, $m \geq 3$ be a cycle in γ . Then in the decomposition of π , it must have either the cycle $(v_1 \ v_2 \ v_3 \dots v_m)$ or $(v_1 \ v_m \ v_{m-1} \dots v_2)$. Note that $(v_1 \ v_2 \ v_3 \dots v_m)^{-1} = (v_1 \ v_m \ v_{m-1} \dots v_2)$.

Let $\pi_\gamma \in f^{-1}(\gamma)$ be fixed. Then $\pi_\gamma = \tau'_1 \tau'_2 \dots \tau'_{k_1} \tau_1 \tau_2 \dots \tau_{k_2}$ where τ'_j is a 2-cycle and τ_j is a m_j -cycle, $m_j \geq 3$. For each $\pi \in f^{-1}(\gamma)$, $\pi = \tau'_1 \tau'_2 \dots \tau'_{k_1} \tau_1^{\pm 1} \tau_2^{\pm 1} \dots \tau_{k_2}^{\pm 1}$. Therefore $\text{sign}(\pi) = \text{sign}(\pi_\gamma)$ and $|f^{-1}(\gamma)| = 2^{k_2}$.

Suppose $\tau'_1 = (u_1 \ u_2)$ (we may assume $u_1 < u_2$). Then $b_{u_1 u_2} b_{u_2 u_1} = w(e_{u_1 u_2}) \overline{w(e_{u_1 u_2})} = |w(e_{u_1 u_2})|^2$ is a term in $\prod_{u=1}^n b_{u\pi(u)}$. Note that $\gamma \setminus C_\gamma$ consists of the union of k_1 edges (K_2) and each of these edges correspond to a τ'_j . Therefore for each $\pi \in f^{-1}(\gamma)$, $|w(\gamma \setminus C_\gamma)|^2$ is a term in $\prod_{u=1}^n b_{u\pi(u)}$.

Suppose $\tau_1 = (v_1 \ v_2 \ v_3 \dots v_m)$. Then $b_{v_1 v_2} b_{v_2 v_3} \dots b_{v_{m-1} v_m} b_{v_m v_1}$ is a term in $\prod_{u=1}^n b_{u\pi(u)}$. Note that C_γ consists of the union of k_2 cycles and each of these cycles correspond to a τ_j . Therefore if we sum up all the $\pi \in f^{-1}(\gamma)$, we have

$$\sum_{\pi \in f^{-1}(\gamma)} \text{sign}(\pi) \prod_{u=1}^n b_{u\pi(u)} = \text{sign}(\pi_\gamma) |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma).$$

Now let us find $\text{sign}(\pi_\gamma)$. A cycle in γ is called an even cycle if it contains odd number of vertices and an odd cycle otherwise. A K_2 in γ is also called an odd cycle. Let the number of even cycles and the number of odd cycles in γ be N_e and N_o respectively. Then $n \equiv N_e \pmod{2}$. Now $\text{sign}(\pi_\gamma) = (-1)^{N_o}$. Since $\text{comp}(\gamma) = N_o + N_e$, we conclude that $\text{sign}(\pi_\gamma) = (-1)^{\text{comp}(\gamma)+n}$. Therefore

$$\sum_{\pi \in f^{-1}(\gamma)} \text{sign}(\pi) \prod_{u=1}^n b_{u\pi(u)} = (-1)^{\text{comp}(\gamma)+n} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma).$$

Hence

$$\begin{aligned}
\det B_{(w,w_1)}(G) &= \sum_{\pi \in S} \text{sign}(\pi) \prod_{u=1}^n b_{u\pi(u)} \\
&= \sum_{\gamma \in \Gamma} \sum_{\pi \in f^{-1}(\gamma)} \text{sign}(\pi) \prod_{u=1}^n b_{u\pi(u)} \\
&= \sum_{\gamma \in \Gamma} (-1)^{\text{comp}(\gamma)+n} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma).
\end{aligned}$$

□

We shall need the following theorem from matrix theory.

Theorem 2.6. ([23, Theorem 7.1.2 on p. 197]) *Let B be a $n \times n$ matrix. Then*

$$\det(xI_n - B) = x^n + \sum_{k=0}^{n-1} (-1)^{n-k} \sum_{1 \leq u_1 < \dots < u_k \leq n} |B(u_1, \dots, u_k; u_1, \dots, u_k)| x^k,$$

where $B(u_1, \dots, u_k; u_1, \dots, u_k)$ is the matrix obtained from B by deleting the u_1, \dots, u_k rows and u_1, \dots, u_k columns. Note that $B(u_1, \dots, u_k; u_1, \dots, u_k)$ is a $(n-k) \times (n-k)$ matrix. □

Lemma 2.7. *Suppose $w_1(u) = 0$ for all $u \in V(G)$. Let Γ_i be the set of all elementary subgraphs of G with $n-i$ vertices and $\phi_{(w,w_1)}(G, x) = \sum_{r=0}^n c_r x^r$, where $n = |V(G)|$. Then $c_n = 1$ and for $i = 0, \dots, n-1$,*

$$c_i = \sum_{\gamma \in \Gamma_i} (-1)^{\text{comp}(\gamma)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma),$$

where C_γ is the union of all the cycles in γ . In particular $c_{n-1} = 0$.

Proof. Let $B = B_{(w,w_1)}(G)$ and $V(G) = \{1, 2, \dots, n\}$. By Theorem 2.6,

$$\phi_{(w,w_1)}(G, x) = \det(xI_n - B) = x^n + \sum_{k=0}^{n-1} (-1)^{n-k} \sum_{1 \leq u_1 < \dots < u_k \leq n} |B(u_1, \dots, u_k; u_1, \dots, u_k)| x^k.$$

If $H(u_1, \dots, u_k) = G \setminus u_1 \dots u_k$ then $B_{(w,w_1)}(H(u_1, \dots, u_k)) = B(u_1, \dots, u_k; u_1, \dots, u_k)$. By Lemma 2.5,

$$\sum_{1 \leq u_1 < \dots < u_k \leq n} |B(u_1, \dots, u_k; u_1, \dots, u_k)| = (-1)^{n-k} \sum_{\gamma \in \Gamma_k} (-1)^{\text{comp}(\gamma)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma).$$

Therefore

$$\phi_{(w,w_1)}(G, x) = x^n + \sum_{k=0}^{n-1} \left(\sum_{\gamma \in \Gamma_k} (-1)^{\text{comp}(\gamma)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma) \right) x^k.$$

Hence the lemma holds. Finally $c_{n-1} = 0$ because Γ_{n-1} is the empty set. □

Lemma 2.8. Suppose $w_1(u) = 0$ for all $u \in V(G)$. Let $\Gamma(c)$ be the set of all elementary subgraphs of G which contains only cycles. Then

$$\phi_{(w,w_1)}(G, x) = \mu_w(G, x) + \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \mu_w(G \setminus C, x).$$

In particular $\phi_{(w,w_1)}(G, x)$ is a polynomial over the field of real number \mathbb{R} .

Proof. Let $|V(G)| = n$ and $\phi_w(G, x) = \sum_{r=0}^n c_r x^r$. By Lemma 2.7, $c_n = 1$ and for $i = 0, \dots, n-1$, $c_i = \sum_{\gamma \in \Gamma_i} (-1)^{\text{comp}(\gamma)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma)$, where Γ_i is the set of all elementary subgraphs of G with $n-i$ vertices and C_γ is the union of all the cycles in γ . Also $c_{n-1} = 0$.

Let $\Gamma_i(1) = \{\gamma \in \Gamma_i : \gamma \text{ does not contain any cycle}\}$ and $\Gamma_i(2) = \Gamma_i \setminus \Gamma_i(1)$. Let $g(\gamma) = (-1)^{\text{comp}(\gamma)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma)$. Then

$$\phi_{(w,w_1)}(G, x) = x^n + \sum_{r=0}^{n-2} \sum_{\gamma \in \Gamma_r(1)} g(\gamma) x^r + \sum_{r=0}^{n-2} \sum_{\gamma \in \Gamma_r(2)} g(\gamma) x^r.$$

Note that

$$\sum_{r=0}^{n-2} \sum_{\gamma \in \Gamma_r(1)} g(\gamma) x^r = \sum_{r=2}^n \sum_{\gamma \in \Gamma_{n-r}(1)} g(\gamma) x^{n-r}.$$

Now if $\gamma \in \Gamma_{n-r}(1)$ then $C_\gamma = \emptyset$ and $\text{comp}(\gamma)$ is the number of K_2 in γ . Therefore $|w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma) = |w(\gamma)|^2$, γ is a $(r/2)$ -matching in G and the number of vertices in γ is $r = 2\text{comp}(\gamma)$. This means that if r is not even then the coefficient of x^{n-r} is zero. Furthermore if $\gamma, \gamma' \in \Gamma_{n-r}(1)$ then $\text{comp}(\gamma) = \text{comp}(\gamma')$. Let $d = \text{comp}(\gamma)$. Then

$$\begin{aligned} \sum_{r=2}^n \sum_{\gamma \in \Gamma_{n-r}(1)} g(\gamma) x^{n-r} &= \sum_{r=2}^n \sum_{\gamma \in \Gamma_{n-r}(1)} (-1)^{\text{comp}(\gamma)} |w(\gamma)|^2 x^{n-r} \\ &= \sum_{d=1}^{\lfloor n/2 \rfloor} (-1)^d \left(\sum_{M \in M_d(G)} |w(M)|^2 \right) x^{n-2d} \end{aligned}$$

and by Lemma 1.1,

$$x^n + \sum_{r=0}^{n-2} \sum_{\gamma \in \Gamma_r(1)} g(\gamma) x^r = \mu_w(G, x).$$

Next

$$\sum_{r=0}^{n-2} \sum_{\gamma \in \Gamma_r(2)} g(\gamma) x^r = \sum_{r=2}^n \sum_{\gamma \in \Gamma_{n-r}(2)} g(\gamma) x^{n-r}.$$

For each $\gamma \in \Gamma_{n-r}(2)$, $C_\gamma \in \Gamma(c)$. We shall partition $\Gamma_{n-r}(2)$ according to $C \in \Gamma(c)$. Let

$$\Gamma_{n-r}(2)(C) = \{\gamma \in \Gamma_{n-r}(2) : \gamma \text{ contains } C \text{ and } \gamma \setminus C \text{ is a disjoint union of } K_2\}.$$

Then $\{\Gamma_{n-r}(2)(C)\}_{C \in \Gamma(c)}$ is a partition for $\Gamma_{n-r}(2)$. If $\gamma \in \Gamma_{n-r}(2)(C)$ then $\text{comp}(\gamma) = \text{comp}(C) + \text{comp}(\gamma \setminus C)$ and the number of vertices in γ is $r = 2\text{comp}(\gamma \setminus C) + |V(C)|$. This means that if

$r \not\equiv |V(C)| \pmod{2}$ then the coefficient of x^{n-r} is zero. Furthermore if $\gamma, \gamma' \in \Gamma_{n-r}(2)(C)$ then $\text{comp}(\gamma \setminus C) = \text{comp}(\gamma' \setminus C)$. So, $\gamma \setminus C$ and $\gamma' \setminus C$ are $((r - |V(C)|)/2)$ -matching in $G \setminus C$. Let $d = \text{comp}(\gamma \setminus C)$. Then by Lemma 1.1,

$$\begin{aligned}
& \sum_{r=2}^n \sum_{\gamma \in \Gamma_{n-r}(2)} g(\gamma) x^{n-r} \\
&= \sum_{r=2}^n \sum_{\gamma \in \Gamma_{n-r}(2)} (-1)^{\text{comp}(\gamma)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma) x^{n-r} \\
&= \sum_{r=2}^n \sum_{C \in \Gamma(c)} \sum_{\gamma \in \Gamma_{n-r}(2)(C)} (-1)^{\text{comp}(\gamma)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma) x^{n-r} \\
&= \sum_{r=2}^n \sum_{C \in \Gamma(c)} \sum_{\gamma \in \Gamma_{n-r}(2)(C)} (-1)^{\text{comp}(C) + \text{comp}(\gamma \setminus C)} |w(\gamma \setminus C_\gamma)|^2 w_2(C_\gamma) x^{n-r} \\
&= \sum_{r=2}^n \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \sum_{\gamma \in \Gamma_{n-r}(2)(C)} (-1)^{\text{comp}(\gamma \setminus C)} |w(\gamma \setminus C)|^2 x^{n-r} \\
&= \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \sum_{r=2}^n \sum_{\gamma \in \Gamma_{n-r}(2)(C)} (-1)^{\text{comp}(\gamma \setminus C)} |w(\gamma \setminus C)|^2 x^{n-r} \\
&= \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \sum_{d=0}^{\lfloor (n-|V(C)|)/2 \rfloor} (-1)^d \left(\sum_{M \in M_d(G \setminus C)} |w(M)|^2 \right) x^{n-|V(C)|-2d} \\
&= \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \mu_w(G \setminus C, x).
\end{aligned}$$

Hence the theorem holds. \square

We wish to show that similar equation (Lemma 2.8) holds even when $w_1(u) \neq 0$ for some $u \in V(G)$. This will be done in Theorem 2.10. Before we do that, let us first prove Lemma 2.9.

Lemma 2.9. *Let $u \in V(G)$. Let G_1 be a graph isomorphic to G . We shall assume $V(G_1) = V(G)$, $E(G_1) = E(G)$ and the weight function (t, t_1) on G_1 is defined by $t(e_{vv'}) = w(e_{vv'})$ for all $e_{vv'} \in E(G_1)$, $t_1(u) = 0$ and $t_1(v) = w_1(v)$ for all $v \in V(G_1) \setminus \{u\}$.*

Let $G_2 = G \setminus u$. Then

$$\eta_{(w, w_1)}(G, x) = \eta_{(t, t_1)}(G_1, x) - w_1(u) \eta_{(w, w_1)}(G_2, x).$$

Proof. Note that

$$\begin{aligned}
\eta_{(w, w_1)}(G, x) &= \sum_{\substack{S \subseteq V(G), \\ u \in S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x) + \\
&\quad \sum_{\substack{S \subseteq V(G), \\ u \notin S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x).
\end{aligned}$$

For each $S \subseteq V(G)$ with $u \in S$, $(-1)^{|V(G \setminus S)|} = (-1)^{|V(G_1 \setminus S)|}$, $w_1(G \setminus S) = t_1(G_1 \setminus S)$ and $H_G(S) = H_{G_1}(S)$. Therefore

$$\sum_{\substack{S \subseteq V(G), \\ u \in S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x) = \sum_{\substack{S \subseteq V(G_1), \\ u \in S}} (-1)^{|V(G_1 \setminus S)|} t_1(G_1 \setminus S) \mu_t(H_{G_1}(S), x).$$

By Lemma 1.7, $\eta_{(t, t_1)}(G_1, x) = \sum_{S \subseteq V(G_1), u \in S} (-1)^{|V(G_1 \setminus S)|} t_1(G_1 \setminus S) \mu_t(H_{G_1}(S), x)$.

For each $S \subseteq V(G)$ with $u \notin S$, $(-1)^{|V(G \setminus S)|} = -(-1)^{|V(G_2 \setminus S)|}$, $w_1(G \setminus S) = w_1(u)w_1(G_2 \setminus S)$ and $H_G(S) = H_{G_2}(S)$. Therefore

$$\begin{aligned} \sum_{\substack{S \subseteq V(G), \\ u \notin S}} (-1)^{|V(G \setminus S)|} w_1(G \setminus S) \mu_w(H_G(S), x) \\ = -w_1(u) \sum_{S \subseteq V(G_2)} (-1)^{|V(G_2 \setminus S)|} w_1(G_2 \setminus S) \mu_w(H_{G_2}(S), x) \\ = -w_1(u) \eta_{(w, w_1)}(G_2, x), \end{aligned}$$

and $\eta_{(w, w_1)}(G, x) = \eta_{(t, t_1)}(G_1, x) - w_1(u) \eta_{(w, w_1)}(G_2, x)$. \square

Theorem 2.10. *Let $\Gamma(c)$ be the set of all elementary subgraphs of G which contains only cycles. Then*

$$\phi_{(w, w_1)}(G, x) = \eta_{(w, w_1)}(G, x) + \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w, w_1)}(G \setminus C, x).$$

In particular $\phi_{(w, w_1)}(G, x)$ is a polynomial over the field of real number \mathbb{R} .

Proof. Let $V(G) = \{1, 2, \dots, n\}$. Let the number of non-zero in the sequence $w_1(1), w_1(2), \dots, w_1(n)$ be denoted by $\kappa(G)$. We shall prove by induction on $\kappa(G)$. If $\kappa(G) = 0$, that is $w_1(j) = 0$ for all j , then the theorem holds (Lemma 1.6 and Lemma 2.8). Suppose $\kappa(G) > 0$. Assume that the theorem holds for all graph G' with $\kappa(G') < \kappa(G)$.

For convenience, we shall assume $w_1(1) \neq 0$ (similar argument can be used if $w_1(u) \neq 0$ for other u). Note that

$$\phi_{(w, w_1)}(G, x) = \det \begin{pmatrix} x - w_1(1) & -w(e_{12}) & \dots & -w(e_{1n}) \\ -\overline{w(e_{12})} & x - w_2(2) & \dots & -w(e_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{w(e_{1n})} & -\overline{w(e_{2n})} & \dots & x - w_1(n) \end{pmatrix}.$$

So by Theorem 1.2.5 on p. 10 of [23],

$$\phi_{(w,w_1)}(G, x) = \det \begin{pmatrix} x & -w(e_{12}) & \dots & -w(e_{1n}) \\ -w(e_{12}) & x - w_1(2) & \dots & -w(e_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ -w(e_{1n}) & -w(e_{2n}) & \dots & x - w_1(n) \end{pmatrix} + \det \begin{pmatrix} -w_1(1) & -w(e_{12}) & \dots & -w(e_{1n}) \\ 0 & x - w_1(2) & \dots & -w(e_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -w(e_{2n}) & \dots & x - w_1(n) \end{pmatrix}.$$

Let G_1 be a graph isomorphic to G . We may assume $V(G_1) = V(G)$ and $E(G_1) = E(G)$. Now let us define the weight function (t, t_1) on G_1 . Set $t(e_{uv}) = w(e_{uv})$ for all $e_{uv} \in E(G_1)$, $t_1(1) = 0$ and $t_1(j) = w_1(j)$ for all $j \geq 2$. Then

$$\phi_{(t,t_1)}(G_1, x) = \det \begin{pmatrix} x & -w(e_{12}) & \dots & -w(e_{1n}) \\ -w(e_{12}) & x - w_1(2) & \dots & -w(e_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ -w(e_{1n}) & -w(e_{2n}) & \dots & x - w_1(n) \end{pmatrix}$$

and by induction (for $\kappa(G_1) < \kappa(G)$),

$$\phi_{(t,t_1)}(G_1, x) = \eta_{(t,t_1)}(G_1, x) + \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(t,t_1)}(G_1 \setminus C, x).$$

Let $G_2 = G \setminus 1$. Then

$$\phi_{(w,w_1)}(G_2, x) = \det \begin{pmatrix} x - w_1(2) & \dots & -w(e_{2n}) \\ \vdots & \ddots & \vdots \\ -w(e_{2n}) & \dots & x - w_1(n) \end{pmatrix}$$

and by induction (for $\kappa(G_2) < \kappa(G)$),

$$\phi_{(w,w_1)}(G_2, x) = \eta_{(w,w_1)}(G_2, x) + \sum_{C \in \Gamma_2(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w,w_1)}(G_2 \setminus C, x),$$

where $\Gamma_2(c)$ is the set of all elementary subgraphs of G_2 which contains only cycles.

Note that $\phi_{(w,w_1)}(G, x) = \phi_{(t,t_1)}(G_1, x) - w_1(1)\phi_{(w,w_1)}(G_2, x)$ and by Lemma 2.9, $\eta_{(w,w_1)}(G, x) = \eta_{(t,t_1)}(G_1, x) - w_1(1)\eta_{(w,w_1)}(G_2, x)$.

Next note that

$$\begin{aligned} \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(t,t_1)}(G_1 \setminus C, x) &= \\ \sum_{\substack{C \in \Gamma(c) \\ 1 \in C}} (-1)^{\text{comp}(C)} w_2(C) \eta_{(t,t_1)}(G_1 \setminus C, x) &+ \\ \sum_{\substack{C \in \Gamma(c) \\ 1 \notin C}} (-1)^{\text{comp}(C)} w_2(C) \eta_{(t,t_1)}(G_1 \setminus C, x). \end{aligned}$$

For each $C \in \Gamma(c)$ with $1 \in C$, we have $G_1 \setminus C = G \setminus C$ (including the weight functions induced by it on the remaining vertices and edges in $G \setminus C$). Therefore $\eta_{(t,t_1)}(G_1 \setminus C, x) = \eta_{(w,w_1)}(G \setminus C, x)$. For each $C \in \Gamma(c)$ with $1 \notin C$, we have $\eta_{(w,w_1)}(G \setminus C, x) = \eta_{(t,t_1)}(G_1 \setminus C, x) - w_1(1)\eta_{(w,w_1)}(G_2 \setminus C, x)$ (Lemma 2.9). Therefore

$$\sum_{\substack{C \in \Gamma(c) \\ 1 \in C}} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w,w_1)}(G \setminus C, x) = \sum_{\substack{C \in \Gamma(c) \\ 1 \in C}} (-1)^{\text{comp}(C)} w_2(C) \eta_{(t,t_1)}(G_1 \setminus C, x),$$

and

$$\begin{aligned} \sum_{\substack{C \in \Gamma(c) \\ 1 \notin C}} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w,w_1)}(G \setminus C, x) = \\ \sum_{\substack{C \in \Gamma(c) \\ 1 \notin C}} (-1)^{\text{comp}(C)} w_2(C) \eta_{(t,t_1)}(G_1 \setminus C, x) - \\ w_1(1) \sum_{C \in \Gamma_2(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w,w_1)}(G_2 \setminus C, x). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w,w_1)}(G \setminus C, x) = \\ \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(t,t_1)}(G_1 \setminus C, x) - \\ w_1(1) \sum_{C \in \Gamma_2(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w,w_1)}(G_2 \setminus C, x), \end{aligned}$$

$$\text{and } \phi_{(w,w_1)}(G, x) = \eta_{(w,w_1)}(G, x) + \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w,w_1)}(G \setminus C, x). \quad \square$$

Example 2.11. Let G and (w, w_1) be as in Example 1.5. Note that the only element in $\Gamma(c)$ is G . Now $\eta_{(w,w_1)}(\emptyset, x) = 1$, $w_2(G) = b_{v_1 v_2} b_{v_2 v_3} b_{v_3 v_1} + b_{v_1 v_3} b_{v_3 v_2} b_{v_2 v_1} = (1 + 2i)(2 - 7i)(-3 - 2i) + (-3 + 2i)(2 + 7i)(1 - 2i) = -108$. By Example 1.5, $\eta_{(w,w_1)}(G, x) = x^3 - 6x^2 - 60x + 88$. So by Theorem 2.10, $\phi_{(w,w_1)}(G, x) = x^3 - 6x^2 - 60x + 88 + (-1)(-108) = x^3 - 6x^2 - 60x + 196$ (see also Example 1.12). \square

The following corollary follows from Theorem 2.10.

Corollary 2.12. *If G is a disjoint union of trees (forest) then $\phi_{(w,w_1)}(G, x) = \eta_{(w,w_1)}(G, x)$.* \square

Note that the converse of Corollary 2.12 is not true in general (see Example 2.13). However if the edge weight function is positive real-valued then it is true (Corollary 2.14).

Example 2.13. Let G be the graph in Figure 5, $V(G) = \{u_1, u_2, u_3, u_4, u_5\}$ and (w, w_1) be as stated. Here we assume $u_1 \equiv 1$, $u_2 \equiv 2$, $u_3 \equiv 3$, $u_4 \equiv 4$ and $u_5 \equiv 5$. Note that

$$\phi_{(w, w_1)}(G, x) = \det \begin{pmatrix} x-2 & -1 & 1-i & 0 & 0 \\ -1 & x-3 & -1 & 0 & 0 \\ 1+i & -1 & x-4 & -1 & -1 \\ 0 & 0 & -1 & x-2 & -1 \\ 0 & 0 & -1 & -1 & x-3 \end{pmatrix},$$

that is $\phi_{(w, w_1)}(G, x) = x^5 - 14x^4 + 70x^3 - 152x^2 + 135x - 35$. Now by using the recurrence in Theorem 2.2, $\eta_{(w, w_1)}(G, x) = x^5 - 14x^4 + 70x^3 - 152x^2 + 135x - 35$. Therefore $\phi_{(w, w_1)}(G, x) = \eta_{(w, w_1)}(G, x)$ but G is not a forest.

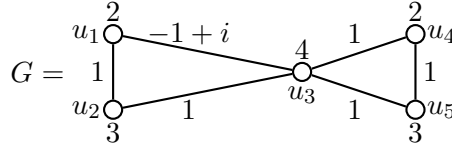


Figure 5.

□

Corollary 2.14. Suppose the edge weight function w of G is positive real-valued. Then G is a disjoint union of trees (forest) if and only if $\phi_{(w, w_1)}(G, x) = \eta_{(w, w_1)}(G, x)$.

Proof. By Corollary 2.12, it is sufficient to prove that if $\phi_{(w, w_1)}(G, x) = \eta_{(w, w_1)}(G, x)$ then G is a forest.

Suppose G is not a forest. By Theorem 2.10,

$$\phi_{(w, w_1)}(G, x) = \eta_{(w, w_1)}(G, x) + \sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w, w_1)}(G \setminus C, x).$$

Therefore $\sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w, w_1)}(G \setminus C, x) = 0$. Let C be the cycle of the least length in G . Suppose there are exactly m cycles of such length. Let it be denoted by C_1, \dots, C_m . Let us look at the coefficient of $x^{n-|C_1|}$. Now the summation over all C_i , $i = 1, \dots, m$, contribute to the coefficient of $x^{n-|C_1|}$. If $C' \in \Gamma(c)$ and $C' \neq C_i$ for all $i = 1, \dots, m$, then it does not contribute to $x^{n-|C_1|}$ because its length is greater and the degree of $\eta_{(w, w_1)}(G \setminus C', x)$ will be less than $x^{n-|C_1|}$ (Lemma 1.8). Each of the C_i contributes exactly $-w_2(C_i) \neq 0$ (by Lemma 2.4, $w_2(C_i) > 0$). Therefore the coefficient of $x^{n-|C_1|}$ is $-\sum_{i=1}^m w_2(C_i) \neq 0$ and $\sum_{C \in \Gamma(c)} (-1)^{\text{comp}(C)} w_2(C) \eta_{(w, w_1)}(G \setminus C, x) \neq 0$, a contradiction. Hence G is a forest. □

Note that Theorem 2.10 and Corollary 2.14 are generalizations of Theorem 4 of [8] and Corollary 4.2 of [8], respectively. This can be seen by taking $w(e) = 1$ for all $e \in E(G)$ and $w_1(u) = 0$ for all $u \in V(G)$, and noting that $\phi_{(w, w_1)}(G, x)$ is the usual characteristic polynomial of the adjacency matrix of G (also together with Lemma 1.3 and Lemma 1.6).

Now let us discuss the ‘ordering’ in $V(G)$. Before we move on to the next two corollaries, it is a good idea to look at Example 1.13 again. Now Corollary 2.15 follows from Corollary 2.12 and Lemma 1.10.

Corollary 2.15. *Let G_1 and G_2 be forests with weight function (w, w_1) and (w', w'_1) , respectively. If G_1 is weight-isomorphic to G_2 , then $\phi_{(w, w_1)}(G_1, x) = \phi_{(w', w'_1)}(G_2, x)$. \square*

Corollary 2.16. *Let G_1 and G_2 be graphs with weight function (w, w_1) and (w', w'_1) , respectively. If G_1 is weight-isomorphic to G_2 and the edge weight functions w, w' take non-zero real number, then $\phi_{(w, w_1)}(G_1, x) = \phi_{(w', w'_1)}(G_2, x)$.*

Proof. If G_1 is a forests then G_2 is also a forests. So we are done by Corollary 2.15. Suppose G_1 is not a forests. Then G_2 is also not a forests. Furthermore every cycle in G_1 is also a cycle in G_2 . Now let us look at the value $w_2(C)$.

Suppose $C = v_1 v_2 \dots v_m v_1$, $m \geq 3$, is a cycle in G_1 . Then

$$\begin{aligned} w_2(C) &= b_{v_1 v_2} b_{v_2 v_3} \dots b_{v_{m-1} v_m} b_{v_m v_1} + b_{v_1 v_m} b_{v_m v_{m-1}} \dots b_{v_3 v_2} b_{v_2 v_1} \\ &= 2b_{v_1 v_2} b_{v_2 v_3} \dots b_{v_{m-1} v_m} b_{v_m v_1} \\ &= 2w(e_{v_1 v_2})w(e_{v_2 v_3}) \dots w(e_{v_{m-1} v_m})w(e_{v_m v_1}) \\ &= 2w(C), \end{aligned}$$

where the second and third equalities follow from the fact that $b_{v_j v_{j+1}} = \overline{b_{v_{j+1} v_j}} = w(e_{v_j v_{j+1}})$ (for the edge weight function w take non-zero real number).

Suppose C is a disjoint union of k cycles C_1, \dots, C_k . Then $w_2(C) = w_2(C_1) \dots w_2(C_k) = 2^k w(C_1) \dots w(C_k) = 2^k w(C)$. So the value of $w_2(C)$ is equal to $2^{\text{comp}(C)}$ times the product of all the weights on the edges in C .

Similarly $w'_2(C)$ is equal to $2^{\text{comp}(C)}$ times the product of all the weights on the edges in C . Therefore $w'_2(C) = w_2(C)$. It then follows from Theorem 2.10 and Lemma 1.10 that $\phi_{(w, w_1)}(G_1, x) = \phi_{(w', w'_1)}(G_2, x)$. \square

The next corollary follows from Corollary 2.12 and the fact that all eigenvalues of a Hermitian matrix are real (see [23, Theorem 7.5.1 on p. 209]).

Corollary 2.17. *If T is a tree then the roots of $\eta_{(w, w_1)}(T, x)$ are real. \square*

Now if $w_1(u) = 0$ for all $u \in V(T)$, we can say further on where it's roots lie. This will done in the next corollary.

Corollary 2.18. *Let T be a tree. Suppose $w_1(u) = 0$ for all $u \in V(T)$. If the maximum valency Δ of T is greater than 1, then the roots of $\eta_{(w, w_1)}(T, x)$ lie in the interval $[-2b_0\sqrt{\Delta-1}, 2b_0\sqrt{\Delta-1}]$, where $b_0 = \max_{e \in E(T)} |w(e)|$.*

Proof. First note that $\phi_{(w, w_1)}(T, x) = \eta_{(w, w_1)}(T, x) = \mu_w(T, x)$ (Corollary 2.12 and Lemma 1.6). Let $B = B_{(w, w_1)}(T) = [b_{uv}]$. Then $b_{uu} = 0$ for all u . Let $b_0 = \max_{u, v} |b_{uv}|$. Then $b_0 = \max_{e \in E(T)} |w(e)|$. Let $C = [c_{uv}]$ where $c_{uv} = 0$ if $b_{uv} = 0$ and $c_{uv} = b_0$ if $b_{uv} \neq 0$. Set $w_0(e) = 1$ for all $e \in E(T)$. Then $C = b_0 B_{(w_0, w_1)}(T)$. Let $B_{(w_0, w_1)}(T) = [d_{uv}]$. Note that $B_{(w_0, w_1)}(T)$ is the adjacency matrix of T .

Now let λ be an eigenvalue of B and $\mathbf{x}_0 = (x_1, \dots, x_n)$ be its corresponding eigenvector. Then

$$|\lambda| = \left| \min_{1 \leq u \leq n, x_u \neq 0} \frac{\sum_{k=1}^n b_{uk} x_k}{x_u} \right| \leq b_0 \min_{1 \leq u \leq n, x_u \neq 0} \frac{\sum_{k=1}^n d_{uk} |x_k|}{|x_u|} \leq b_0 r,$$

where r is a positive eigenvalue of $B_{(w_0, w_1)}(T)$ for which the absolute value of any eigenvalues of $B_{(w_0, w_1)}(T)$ is at most r (see the discussion on p. 534, Proposition 2 on p. 535 and Theorem 1 on p. 536 of [20]). By Theorem 6.3 on p. 87 of [5], $r \leq 2\sqrt{\Delta - 1}$. Hence $|\lambda| \leq 2b_0\sqrt{\Delta - 1}$. \square

3 The Path-tree

The notion of a path-tree of a graph was first introduced by Godsil [6, Section 2] (see also [5, Section 6.1]). Let G be a graph with a vertex u . The *path-tree* $T(G, u)$ is the tree with the paths in G starting at u as its vertices, and two such paths are joined by an edge if one is a maximal subpath of the other. The vertex u is itself a path, and so it is a vertex of $T(G, u)$ and will also be denoted by u . Now let us assign the weight to $T(G, u)$. Note that two vertices, p_1 and p_2 in $V(T(G, u))$ are joined by an edge if and only if $p_1 = p_2uv$ or $p_2 = p_1uv$ for some edge $e_{uv} \in E(G)$. We set $w^T(e_{p_1p_2}) = w(e_{uv})$.

Let p be a vertex in $T(G, u)$. If $p = u$, we set $w_1^T(p) = w_1(u)$. If p is a path with length at least 1, then we set $w_1^T(p) = w_1(v)$, where $v \neq u$ is an endpoint of p .

So for each graph G and weight function (w, w_1) , there corresponds a path-tree $T(G, u)$ and weight function (w^T, w_1^T) . For convenience, when there is no confusion, we shall write w^T as w and w_1^T as w_1 .

Note that if G is a tree, then G is weight-isomorphic to $T(G, u)$. This can be seen by part (b) of Lemma 2.4 of [6] and keeping track of the weights on the edges and vertices.

The next theorem is a generalization of Theorem 2.5 of [6]. However its proof is similar to that in [6]. In fact it can be proved by using Lemma 2.4 of [6] and Theorem 2.2. The details of the proof are omitted.

Theorem 3.1. *Let u be a vertex in G and $T := T(G, u)$ be the path-tree of G with respect to u . Then*

$$\frac{\eta_{(w, w_1)}(G \setminus u, x)}{\eta_{(w, w_1)}(G, x)} = \frac{\eta_{(w, w_1)}(T \setminus u, x)}{\eta_{(w, w_1)}(T, x)}.$$

\square

The next corollary follows easily from Theorem 3.1. For the sake of completeness, we shall give a proof.

Corollary 3.2. *Let G be a connected graph and u be a vertex in G . Let $T = T(G, u)$ be the path tree of G . Then $\eta_{(w, w_1)}(G, x)$ divides $\eta_{(w, w_1)}(T, x)$.*

Proof. If G is a tree then by part (b) of Lemma 2.4 of [6], we deduce that G is weight-isomorphic to T . It then follows from Lemma 1.10 that $\eta_{(w, w_1)}(G, x) = \eta_{(w, w_1)}(T, x)$. Hence the corollary holds. We may assume inductively that the corollary holds for all connected subgraphs of G . Let $G \setminus u = H_1 \cup \dots \cup H_k$ where H_1, \dots, H_k are components of $G \setminus u$. Then by part (a) of Theorem 2.2,

$$\eta_{(w, w_1)}(G \setminus u, x) = \prod_{j=1}^k \eta_{(w, w_1)}(H_j, x).$$

For each j , let $v_j \in V(H_j)$ be such that e_{uv_j} is an edge in G . By part (c) of Lemma 2.4 of [6], we deduce that (keeping track of the weights) the component of $T(G, u) \setminus u$ that contains the vertex

$p_0 = uv_j$ is isomorphic to the path tree $T(H_j, v_j)$. Note also that $T(G \setminus u, v_j) = T(H_j, v_j)$. Therefore by part (a) of Theorem 2.2, we deduce that $\prod_{j=1}^k \eta_{(w, w_1)}(T(G \setminus u, v_j), x)$ divides $\eta_{(w, w_1)}(T(G, u) \setminus u, x)$. By induction hypothesis, $\eta_{(w, w_1)}(H_j, x)$ divides $\eta_{(w, w_1)}(T(H_j, v_j), x)$. Therefore $\eta_{(w, w_1)}(G \setminus u, x)$ divides $\eta_{(w, w_1)}(T(G, u) \setminus u, x)$. By Theorem 3.1,

$$\frac{\eta_{(w, w_1)}(G \setminus u, x)}{\eta_{(w, w_1)}(G, x)} = \frac{\eta_{(w, w_1)}(T \setminus u, x)}{\eta_{(w, w_1)}(T, x)}.$$

Hence $\eta_{(w, w_1)}(G, x)$ divides $\eta_{(w, w_1)}(T, x)$. \square

The following two corollaries follows from part (a) of Theorem 2.2, Corollary 3.2, Corollary 2.17 and Corollary 2.18.

Corollary 3.3. *Let G be a graph. Then the roots of $\eta_{(w, w_1)}(G, x)$ are real.* \square

Corollary 3.4. *Let G be a graph. Suppose $w_1(u) = 0$ for all $u \in V(G)$. If the maximum valency Δ of G is greater than 1, then the roots of $\eta_{(w, w_1)}(G, x)$ lie in the interval $[-2b_0\sqrt{\Delta-1}, 2b_0\sqrt{\Delta-1}]$, where $b_0 = \max_{e \in E(G)} |w(e)|$.* \square

4 Vertex classification

The following lemma can be deduced using equation (2) on p. 29 of [5] (see the proof of Theorem 5.3 on p. 29 of [5] for the details).

Lemma 4.1. *Let $B = [b_{uv}]$ be an $n \times n$ Hermitian matrix. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be all the eigenvalues of B with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Let $\theta_1, \theta_2, \dots, \theta_{n-1}$ be all the eigenvalues of $B(u; u)$ with $\theta_1 \geq \theta_2 \geq \dots \geq \theta_{n-1}$ ($B(u; u)$ is the matrix obtained from B by deleting the u row and the u column). Then*

$$\lambda_1 \geq \theta_1 \geq \lambda_2 \geq \theta_2 \geq \dots \geq \lambda_{n-1} \geq \theta_{n-1} \geq \lambda_n.$$

\square

Lemma 4.2. *Let $u \in V(G)$ and θ be a real number. Then*

$$\text{mult}(\theta, G, \eta_{(w, w_1)}) - 1 \leq \text{mult}(\theta, G \setminus u, \eta_{(w, w_1)}) \leq \text{mult}(\theta, G, \eta_{(w, w_1)}) + 1.$$

Proof. By Theorem 3.1,

$$\frac{\eta_{(w, w_1)}(G \setminus u, x)}{\eta_{(w, w_1)}(G, x)} = \frac{\eta_{(w, w_1)}(T \setminus u, x)}{\eta_{(w, w_1)}(T, x)}.$$

Let $\eta_{(w, w_1)}(G \setminus u, x)/\eta_{(w, w_1)}(G, x) = (x - \theta)^r h(x)/g(x)$ where $h(x)$ and $g(x)$ are polynomials such that $h(\theta) \neq 0 \neq g(\theta)$ and $r = \text{mult}(\theta, G \setminus u, \eta_{(w, w_1)}) - \text{mult}(\theta, G, \eta_{(w, w_1)})$.

By Corollary 2.12, $\phi_{(w, w_1)}(T, x) = \eta_{(w, w_1)}(T, x)$ and $\phi_{(w, w_1)}(T \setminus u, x) = \eta_{(w, w_1)}(T \setminus u, x)$. Let $\text{mult}(\theta, T, \eta_{(w, w_1)}) = m$. By Lemma 4.1, we deduce that $m - 1 \leq \text{mult}(\theta, T \setminus u, \eta_{(w, w_1)}) \leq m + 1$. Now $\eta_{(w, w_1)}(T \setminus u, x)/\eta_{(w, w_1)}(T, x) = (x - \theta)^r h(x)/g(x)$ with $r = \text{mult}(\theta, T \setminus u, \eta_{(w, w_1)}) - \text{mult}(\theta, T, \eta_{(w, w_1)})$. So, $-1 \leq r \leq 1$ and the lemma holds. \square

The following definition is motivated by Lemma 4.2 and followed Godsil's approach [7, Section 3]

Definition 4.3. For any $u \in V(G)$,

- (a) u is (θ, w, w_1) -essential if $\text{mult}(\theta, G \setminus u, \eta_{(w, w_1)}) = \text{mult}(\theta, G, \eta_{(w, w_1)}) - 1$,
- (b) u is (θ, w, w_1) -neutral if $\text{mult}(\theta, G \setminus u, \eta_{(w, w_1)}) = \text{mult}(\theta, G, \eta_{(w, w_1)})$,
- (c) u is (θ, w, w_1) -positive if $\text{mult}(\theta, G \setminus u, \eta_{(w, w_1)}) = \text{mult}(\theta, G, \eta_{(w, w_1)}) + 1$.

Furthermore if u is not (θ, w, w_1) -essential but it is adjacent to some (θ, w, w_1) -essential vertex, we say u is (θ, w, w_1) -special. A graph G is said to be (θ, w, w_1) -critical if all vertices in G are (θ, w, w_1) -essential and $\text{mult}(\theta, G, \eta_{(w, w_1)}) = 1$.

The following lemma is a generalization of [7, Lemma 3.1]. However, its proof is similar to that in [7]. In fact we just need to compare the multiplicity of θ as a root on both sides on the equation in part (d) of Theorem 2.2. The details are omitted.

Lemma 4.4. For any graph G , it has at least one (θ, w, w_1) -essential vertex provided that θ is a root of $\eta_{(w, w_1)}(G, x)$. \square

The next lemma is a generalization of the Heilmann-Lieb equation [7, Lemma 2.4] (see also [9, Theorem 6.3] and [5, Lemma 4.1 on p. 104]). This can be seen by taking $w(e) = 1$ for all $e \in E(G)$ and $w_1(u) = 0$ for all $u \in V(G)$ (also together with Lemma 1.3 and Lemma 1.6). It can be proved by using induction on the number of edges and Theorem 2.2 (following a similar argument as in [5]).

Lemma 4.5. Let $u, v \in V(G)$ and $u \neq v$. Then

$$\eta_{(w, w_1)}(G \setminus u, x) \eta_{(w, w_1)}(G \setminus v, x) - \eta_{(w, w_1)}(G, x) \eta_{(w, w_1)}(G \setminus uv, x) = \sum_{p \in P_{uv}(G)} (|w(p)| \eta_{(w, w_1)}(G \setminus p, x))^2, \quad (*)$$

where $P_{uv}(G)$ is the set of all the paths in G that have u and v as endpoints. \square

The next corollary is a generalization of [7, Corollary 2.5]. It is a consequence of Lemma 4.5 and Lemma 4.2 (following a similar argument as in [7]). The details are omitted.

Corollary 4.6. Let p be a path of length at least 1 in G . Then

$$\text{mult}(\theta, G \setminus p, \eta_{(w, w_1)}) \geq \text{mult}(\theta, G, \eta_{(w, w_1)}) - 1.$$

\square

Corollary 4.7. Let G be a graph. Then

- (a) the maximum multiplicity of a root of $\eta_{(w, w_1)}(G, x)$ is at most equal to the number of vertex-disjoint paths required to cover G ,
- (b) the number of distinct roots of $\eta_{(w, w_1)}(G, x)$ is at least equal to the number of vertices in the longest path in G .

Proof. (a) Let p_1, \dots, p_k be all the vertex-disjoint paths that cover G . By Corollary 4.6 and Lemma 4.2 (in the case if p_j is a single vertex), we have $\text{mult}(\theta, G \setminus p_1, \eta_{(w, w_1)}) \geq \text{mult}(\theta, G, \eta_{(w, w_1)}) - 1$ and inductively $\text{mult}(\theta, G \setminus (p_1 \cup \dots \cup p_{k-1} \cup p_k), \eta_{(w, w_1)}) \geq \text{mult}(\theta, G, \eta_{(w, w_1)}) - k$. Note that $\text{mult}(\theta, G \setminus (p_1 \cup \dots \cup p_{k-1} \cup p_k), \eta_{(w, w_1)}) = 0$. Hence $\text{mult}(\theta, G, \eta_{(w, w_1)}) \leq k$.

(b) Let p be a path in G . Let Θ be the set of all distinct roots of $\eta_{(w, w_1)}(G, x)$. Now $\text{mult}(\theta, G \setminus p, \eta_{(w, w_1)}) \geq \text{mult}(\theta, G, \eta_{(w, w_1)}) - 1$ (Corollary 4.6 and Lemma 4.2) implies that

$$\sum_{\theta \in \Theta} \text{mult}(\theta, G \setminus p, \eta_{(w, w_1)}) \geq \sum_{\theta \in \Theta} \text{mult}(\theta, G, \eta_{(w, w_1)}) - |\Theta|.$$

Since $\sum_{\theta \in \Theta} \text{mult}(\theta, G \setminus p, \eta_{(w, w_1)}) = V(G \setminus p)$ and $\sum_{\theta \in \Theta} \text{mult}(\theta, G, \eta_{(w, w_1)}) = V(G)$, we have $|\Theta| \geq |V(p)|$. \square

Definition 4.8. A path p in G is said to be (θ, w, w_1) -essential if

$$\text{mult}(\theta, G \setminus p, \eta_{(w, w_1)}) = \text{mult}(\theta, G, \eta_{(w, w_1)}) - 1.$$

So if a path q is not (θ, w, w_1) -essential, then $\text{mult}(\theta, G \setminus q, \eta_{(w, w_1)}) \geq \text{mult}(\theta, G, \eta_{(w, w_1)})$ (Corollary 4.6).

Part (a) and (b) of the next lemma are generalizations of [7, Lemma 3.2] and [7, Lemma 3.3], respectively. However their proofs are similar to that in [7]. In fact for part (a), it can be deduced from part (c) of Theorem 2.2 and Corollary 4.6, whereas for part (b), it can be deduced from Lemma 4.5 and Lemma 4.2. The details are omitted.

Lemma 4.9. Let G be a graph and θ be a root of $\eta_{(w, w_1)}(G, x)$. Then

- (a) for any (θ, w, w_1) -essential vertex u with $\theta \neq w_1(u)$, there is a vertex v such that the path $p = uv$ is (θ, w, w_1) -essential,
- (b) if u is not (θ, w, w_1) -essential, then for any path p that ends with u , p is not (θ, w, w_1) -essential. \square

5 Gallai-Edmonds decomposition

We shall begin by showing that a (θ, w, w_1) -special vertex is (θ, w, w_1) -positive. This is a generalization of Corollary 4.3 of [7].

Lemma 5.1. Let $u \in V(G)$. If u is (θ, w, w_1) -special then u is (θ, w, w_1) -positive.

Proof. By Definition 4.3, there is a (θ, w, w_1) -essential vertex v such that $e_{uv} \in E(G)$. Since u is not (θ, w, w_1) -essential, by part (b) of Lemma 4.9, the path $p = uv$ is not (θ, w, w_1) -essential. By Corollary 4.6 (see also Definition 4.8), $\text{mult}(\theta, G \setminus uv, \eta_{(w, w_1)}) = \text{mult}(\theta, G \setminus p, \eta_{(w, w_1)}) \geq k$ where $k = \text{mult}(\theta, G, \eta_{(w, w_1)})$. Also u is either (θ, w, w_1) -positive or (θ, w, w_1) -neutral (Lemma 4.2).

Suppose u is (θ, w, w_1) -neutral. Then $\text{mult}(\theta, G \setminus u, \eta_{(w, w_1)}) = k$. Now the multiplicity of θ as a root of $\eta_{(w, w_1)}(G \setminus u, x)\eta_{(w, w_1)}(G \setminus v, x)$ is exactly $2k - 1$ and the multiplicity of θ as a root of $\eta_{(w, w_1)}(G, x)\eta_{(w, w_1)}(G \setminus uv, x)$ is at least $2k$. This implies that the multiplicity of θ as a root of $\sum_{p \in P_{uv}(G)} (|w(p)|\eta_{(w, w_1)}(G \setminus p, x))^2$ is $2k - 1$ (Lemma 4.5) which is a contradiction since the multiplicity of θ as a root of $\sum_{p \in P_{uv}(G)} (|w(p)|\eta_{(w, w_1)}(G \setminus p, x))^2$ is at least $2k$. Hence u is (θ, w, w_1) -positive. \square

By Definition 4.3 and Lemma 5.1, we have

$$V(G) = D_{(\theta, w, w_1)}(G) \cup A_{(\theta, w, w_1)}(G) \cup P_{(\theta, w, w_1)}(G) \cup N_{(\theta, w, w_1)}(G),$$

where

$D_{(\theta, w, w_1)}(G)$ is the set of all (θ, w, w_1) -essential vertices in G ,

$A_{(\theta, w, w_1)}(G)$ is the set of all (θ, w, w_1) -special vertices in G ,

$N_{(\theta, w, w_1)}(G)$ is the set of all (θ, w, w_1) -neutral vertices in G ,

$P_{(\theta, w, w_1)}(G) = Q_{(\theta, w, w_1)}(G) \setminus A_{(\theta, w, w_1)}(G)$, where $Q_{(\theta, w, w_1)}(G)$ is the set of all (θ, w, w_1) -positive vertices in G ,

is a partition of $V(G)$.

The following lemma is a generalization of Theorem 4.2 of [7] and its proof is similar to that in [7]. In fact for part (a), it can be deduced from Lemma 4.2, whereas for part (b) and (c), it can be deduced by comparing the multiplicity of θ as a root in the equation (*) of Lemma 4.5. The details are omitted.

Lemma 5.2. *Let $\text{mult}(\theta, G, \eta_{(w, w_1)}) = k$ and $v \in V(G)$ be (θ, w, w_1) -positive. Then*

- (a) *if u is (θ, w, w_1) -essential in G then it is (θ, w, w_1) -essential in $G \setminus v$,*
- (b) *if u is (θ, w, w_1) -positive in G then it is (θ, w, w_1) -essential or (θ, w, w_1) -positive in $G \setminus v$,*
- (c) *if u is (θ, w, w_1) -neutral in G then it is (θ, w, w_1) -essential or (θ, w, w_1) -neutral in $G \setminus v$. \square*

The following lemma can be proved similarly by comparing the multiplicity of θ on both sides of (*) of Lemma 4.5 (see [3, Proposition 2.9]).

Lemma 5.3. *Let $\text{mult}(\theta, G, \eta_{(w, w_1)}) = k$ and $v \in V(G)$ be (θ, w, w_1) -neutral. Then*

- (a) *if u is (θ, w, w_1) -essential in G then it is (θ, w, w_1) -essential in $G \setminus v$,*
- (b) *if u is (θ, w, w_1) -positive in G then it is either (θ, w, w_1) -positive or (θ, w, w_1) -neutral in $G \setminus v$,*
- (c) *if u is (θ, w, w_1) -neutral in G then it is either (θ, w, w_1) -positive or (θ, w, w_1) -neutral in $G \setminus v$. \square*

Lemma 5.4. *Let v, z be (θ, w, w_1) -essential in G and $\text{mult}(\theta, G \setminus vz, \eta_{(w, w_1)}) \geq \text{mult}(\theta, G, \eta_{(w, w_1)}) - 1$. If p is a path in G with endpoints v and z , then p is (θ, w, w_1) -essential in G .*

Proof. Note that the multiplicity of θ as a root of $\eta_{(w, w_1)}(G \setminus z, x)\eta_{(w, w_1)}(G \setminus v, x)$ is $2k - 2$, where $k = \text{mult}(\theta, G, \eta_{(w, w_1)})$. Also the multiplicity of θ as a root of $\eta_{(w, w_1)}(G, x)\eta_{(w, w_1)}(G \setminus vz, x)$ is at least $2k - 1$ (for $\text{mult}(\theta, G \setminus vz, \eta_{(w, w_1)}) \geq k - 1$). This implies that the multiplicity of θ as a root of $\sum_{q \in P_{vz}(G)} (|w(q)|\eta_{(w, w_1)}(G \setminus q, x))^2$ is $2k - 2$ (Lemma 4.5). Thus $\text{mult}(\theta, G \setminus q, \eta_{(w, w_1)}) = k - 1$ for all $q \in P_{vz}(G)$; in particular $\text{mult}(\theta, G \setminus p, \eta_{(w, w_1)}) = k - 1$, i.e. p is (θ, w, w_1) -essential (Definition 4.8). \square

The next lemma is somewhat similar to [3, Lemma 4.1]. Basically it is the essence of [3, Lemma 4.1, Lemma 4.2 and Lemma 4.3].

Lemma 5.5. *Let $u, v, z \in V(G)$ be such that u is adjacent to v and z . Suppose u is (θ, w, w_1) -special and v is (θ, w, w_1) -essential in G . Let $G' = G - e_{uz}$. Then $\text{mult}(\theta, G', \eta_{(w, w_1)}) = \text{mult}(\theta, G, \eta_{(w, w_1)})$, u is (θ, w, w_1) -positive in G' . Furthermore if the path $p = vuz$ is not (θ, w, w_1) -essential in G , then u is (θ, w, w_1) -special in G' .*

Proof. Let $\text{mult}(\theta, G, \eta_{(w, w_1)}) = k$. Then $\text{mult}(\theta, G \setminus u, \eta_{(w, w_1)}) = k + 1$ and $\text{mult}(\theta, G \setminus v, \eta_{(w, w_1)}) = k - 1$. It is not hard to deduce from Lemma 4.2, Lemma 5.1 and part (a) of Lemma 5.2, that $\text{mult}(\theta, G \setminus uz, \eta_{(w, w_1)}) \geq k$, $\text{mult}(\theta, G \setminus uv, \eta_{(w, w_1)}) = k$.

Now $\text{mult}(\theta, G' \setminus u, \eta_{(w, w_1)}) = \text{mult}(\theta, G \setminus u, \eta_{(w, w_1)}) = k + 1$ (for $G' \setminus u = G \setminus u$) implies that $\text{mult}(\theta, G', \eta_{(w, w_1)}) = k, k + 1$ or $k + 2$ (Lemma 4.2).

Suppose $\text{mult}(\theta, G', \eta_{(w, w_1)}) = k + 2$. Then by Corollary 4.6, $\text{mult}(\theta, G' \setminus uv, \eta_{(w, w_1)}) \geq k + 1$, a contradiction (for $\text{mult}(\theta, G' \setminus uv, \eta_{(w, w_1)}) = \text{mult}(\theta, G \setminus uv, \eta_{(w, w_1)}) = k$).

Suppose $\text{mult}(\theta, G', \eta_{(w, w_1)}) = k + 1$. Then u is (θ, w, w_1) -neutral in G' (for $\text{mult}(\theta, G' \setminus u, \eta_{(w, w_1)}) = k + 1$). By part (b) of Lemma 4.9, the path uv is not (θ, w, w_1) -essential in G' . It then follows from Corollary 4.6 and Definition 4.8, that $\text{mult}(\theta, G' \setminus uv, \eta_{(w, w_1)}) \geq k + 1$, a contradiction (for $\text{mult}(\theta, G' \setminus uv, \eta_{(w, w_1)}) = k$).

So $\text{mult}(\theta, G', \eta_{(w, w_1)}) = k$ and u is (θ, w, w_1) -positive in G' .

Suppose $p = vuz$ is not (θ, w, w_1) -essential in G . Then by Definition 4.8, $\text{mult}(\theta, G \setminus vuz, \eta_{(w, w_1)}) \geq k$. Now by part (b) of Theorem 2.2, $\eta_{(w, w_1)}(G \setminus v, x) = \eta_{(w, w_1)}(G' \setminus v, x) - |w(e_{uz})|^2 \eta_{(w, w_1)}(G \setminus vuz, x)$. Since $\text{mult}(\theta, G \setminus v, \eta_{(w, w_1)}) = k - 1$, we deduce that $\text{mult}(\theta, G' \setminus v, \eta_{(w, w_1)}) = k - 1$. Hence v is (θ, w, w_1) -essential and u is (θ, w, w_1) -special in G' (for u is adjacent to v and u is not (θ, w, w_1) -essential in G'). \square

The next lemma is a generalization of [3, Proposition 5.1] and it can be proved using similar argument as in [3]. Nevertheless we shall give the details.

Lemma 5.6. *Let u be (θ, w, w_1) -special in G and v be (θ, w, w_1) -essential in $G \setminus u$. Then v is either (θ, w, w_1) -positive or (θ, w, w_1) -essential in G .*

Proof. Let $\text{mult}(\theta, G, \eta_{(w, w_1)}) = k$. Then $\text{mult}(\theta, G \setminus uv, \eta_{(w, w_1)}) = k$ (Lemma 5.1). Suppose v is (θ, w, w_1) -neutral in G . Then u is (θ, w, w_1) -neutral in $G \setminus v$. But u is adjacent to a (θ, w, w_1) -essential vertex z in G , so by part (a) of Lemma 5.3, z is (θ, w, w_1) -essential in $G \setminus v$, which means that u is (θ, w, w_1) -special and thus (θ, w, w_1) -positive in $G \setminus v$ (Lemma 5.1) a contradiction. Hence v is either (θ, w, w_1) -positive or (θ, w, w_1) -essential in G . \square

A vertex is said to be an *isolated vertex* in G if it is not adjacent to any other vertices in G .

Lemma 5.7. *Let u be an isolated vertex in G . Then*

- (a) *if $\theta = w_1(u)$ then u is (θ, w, w_1) -essential in G ,*
- (b) *if $\theta \neq w_1(u)$ then u is (θ, w, w_1) -neutral in G .*

Proof. The lemma follows by comparing the multiplicity of θ as a root of both sides of the equation $\eta_{(w,w_1)}(G, x) = (x - w_1(u)) \eta_{(w,w_1)}(G \setminus u, x)$ (part (c) of Theorem 2.2). \square

The following fact was first observed by Chen and Ku in their proof of [3, Theorem 1.5] for the classification of vertices, using the root of the usual matching polynomial $\mu(G, x)$. We shall give the details of the proof.

Lemma 5.8. *Let u be (θ, w, w_1) -special in G . Then the degree of u is at least two.*

Proof. Suppose the contrary. Then the degree of u is one and the vertex adjacent to u , say z is (θ, w, w_1) -essential. Now $\text{mult}(\theta, G \setminus u, \eta_{(w,w_1)}) = k + 1$ (Lemma 5.1), $\text{mult}(\theta, G \setminus z, \eta_{(w,w_1)}) = k - 1$, where $k = \text{mult}(\theta, G, \eta_{(w,w_1)})$. Also by part (a) of Lemma 5.2, $\text{mult}(\theta, G \setminus uz, \eta_{(w,w_1)}) = k$.

Let $G' = G - e_{uz}$. From $\eta_{(w,w_1)}(G, x) = \eta_{(w,w_1)}(G', x) - |w(e_{uz})|^2 \eta_{(w,w_1)}(G \setminus uz, x)$ (part (b) of Theorem 2.2), we deduce that $\text{mult}(\theta, G', \eta_{(w,w_1)}) \geq k$. On the other hand, $\text{mult}(\theta, G' \setminus z, \eta_{(w,w_1)}) = \text{mult}(\theta, G \setminus z, \eta_{(w,w_1)}) = k - 1$ (for $G' \setminus z = G \setminus z$) implies that $\text{mult}(\theta, G', \eta_{(w,w_1)}) = k - 2, k - 1$ or k (Lemma 4.2). Hence $\text{mult}(\theta, G', \eta_{(w,w_1)}) = k$.

Now $\text{mult}(\theta, G' \setminus u, \eta_{(w,w_1)}) = \text{mult}(\theta, G \setminus u, \eta_{(w,w_1)}) = k + 1$, that is u is (θ, w, w_1) -positive in G' . But this contradicts Lemma 5.7 (for u is an isolated vertex in G'). Hence the degree of u is at least two. \square

Lemma 5.9. *Let G be the union of two graphs G_1 and G_2 . Let $u \in G_1$ and $v \in G_2$. Then v is (θ, w, w_1) -essential in G if and only if it is (θ, w, w_1) -essential in $G \setminus u$.*

Proof. By part (a) of Theorem 2.2, we deduce that

$$\text{mult}(\theta, G, \eta_{(w,w_1)}) = \text{mult}(\theta, G_1, \eta_{(w,w_1)}) + \text{mult}(\theta, G_2, \eta_{(w,w_1)}),$$

$$\text{mult}(\theta, G \setminus u, \eta_{(w,w_1)}) = \text{mult}(\theta, G_1 \setminus u, \eta_{(w,w_1)}) + \text{mult}(\theta, G_2, \eta_{(w,w_1)}),$$

$$\text{mult}(\theta, G \setminus v, \eta_{(w,w_1)}) = \text{mult}(\theta, G_1, \eta_{(w,w_1)}) + \text{mult}(\theta, G_2 \setminus v, \eta_{(w,w_1)}),$$

$$\text{mult}(\theta, G \setminus uv, \eta_{(w,w_1)}) = \text{mult}(\theta, G_1 \setminus u, \eta_{(w,w_1)}) + \text{mult}(\theta, G_2 \setminus v, \eta_{(w,w_1)}).$$

Suppose v is (θ, w, w_1) -essential in G . Then $\text{mult}(\theta, G \setminus v, \eta_{(w,w_1)}) = \text{mult}(\theta, G, \eta_{(w,w_1)}) - 1$. This implies that $\text{mult}(\theta, G_2 \setminus v, \eta_{(w,w_1)}) = \text{mult}(\theta, G_2, \eta_{(w,w_1)}) - 1$, and thus $\text{mult}(\theta, G \setminus uv, \eta_{(w,w_1)}) = \text{mult}(\theta, G \setminus u, \eta_{(w,w_1)}) - 1$. Hence v is (θ, w, w_1) -essential in $G \setminus u$. The converse is proved similarly. \square

For the proof of the following lemma, we shall use similar ideas as in [3, Theorem 1.5], that is by using edge manipulation and assuming first that the special vertex is of degree two. However, we cannot use the same argument as in [3] directly, because Chen and Ku assumed that $\theta \neq 0$ in their proof (in our case this is equivalent to $\theta \neq w_1(u)$ where u is the (θ, w, w_1) -special vertex).

Lemma 5.10. *Let u be (θ, w, w_1) -special in G and the degree of u is two. Then v is (θ, w, w_1) -essential in $G \setminus u$ if and only if v is (θ, w, w_1) -essential in G .*

Proof. Let $\text{mult}(\theta, G, \eta_{(w, w_1)}) = k$. Suppose v is (θ, w, w_1) -essential in $G \setminus u$. Then by Lemma 5.1 $\text{mult}(\theta, G \setminus uv, \eta_{(w, w_1)}) = k$. By Lemma 5.6, v is either (θ, w, w_1) -positive or (θ, w, w_1) -essential in G . Suppose v is (θ, w, w_1) -positive in G . We shall show that this cannot happen.

Let z_1, z_2 be the two vertices adjacent to u . Without loss of generality, we assume z_1 is (θ, w, w_1) -essential in G . First note that $\text{mult}(\theta, G \setminus v, \eta_{(w, w_1)}) = k + 1$ and $\text{mult}(\theta, G \setminus vuz_2, \eta_{(w, w_1)}) \geq k$ (Corollary 4.6). Let $G' = G - e_{uz_2}$. Now we show that z_2 is (θ, w, w_1) -essential and the path $p = z_1uz_2$ is (θ, w, w_1) -essential in G . Suppose $p = z_1uz_2$ is not (θ, w, w_1) -essential in G . Then by Lemma 5.5, u is (θ, w, w_1) -special G' , a contrary to Lemma 5.8 (for u is of degree one in G'). Hence the path $p = z_1uz_2$ is (θ, w, w_1) -essential in G . By part (b) of Lemma 4.9, z_2 is (θ, w, w_1) -essential in G . This also means that $v \neq z_1, z_2$ (for we assume v to be (θ, w, w_1) -positive in G).

Now we show that $\text{mult}(\theta, G \setminus z_1z_2, \eta_{(w, w_1)}) \geq k - 1$. Note that $\text{mult}(\theta, G \setminus z_1, \eta_{(w, w_1)}) = k - 1$. So by Lemma 4.2, $\text{mult}(\theta, G \setminus z_1z_2, \eta_{(w, w_1)}) = k - 2, k - 1$ or k . Suppose $\text{mult}(\theta, G \setminus z_1z_2, \eta_{(w, w_1)}) = k - 2$. Note that u is an isolated vertex in $G \setminus z_1z_2$. So $\text{mult}(\theta, G \setminus z_1z_2u, \eta_{(w, w_1)}) \leq k - 2$ (Lemma 5.7). But this contradicts the conclusion of the previous paragraph that $\text{mult}(\theta, G \setminus z_1uz_2, \eta_{(w, w_1)}) = k - 1$ (the path $p = z_1uz_2$ is (θ, w, w_1) -essential in G). Hence $\text{mult}(\theta, G \setminus z_1z_2, \eta_{(w, w_1)}) \geq k - 1$.

Now we show that $p = z_1uz_2$ is the only path with endpoints z_1 and z_2 in G . Suppose the contrary. Then there exists a path $q_1 \neq p$ with endpoints z_1 and z_2 . Note that q_1 does not contain the vertex u . By Lemma 5.4, q_1 is (θ, w, w_1) -essential in G , that is $\text{mult}(\theta, G \setminus q_1, \eta_{(w, w_1)}) = k - 1$. Since u is an isolated vertex in $G \setminus q_1$, by Lemma 5.7, $\text{mult}(\theta, G \setminus uq_1, \eta_{(w, w_1)}) \leq k - 1$. Since q_1 is a path that begins with z_1 and ends with z_2 , uq_1 is a path that begins with u and ends with z_2 . But by part (b) of Lemma 4.9, $\text{mult}(\theta, G \setminus uq_1, \eta_{(w, w_1)}) \geq k$, a contradiction. Hence $p = z_1uz_2$ is the only path with endpoints z_1 and z_2 in G .

Recall that $G' = G - e_{uz_2}$. So u is (θ, w, w_1) -positive in G' and $\text{mult}(\theta, G', \eta_{(w, w_1)}) = k$ (Lemma 5.5). Next $\text{mult}(\theta, G' \setminus z_2, \eta_{(w, w_1)}) = \text{mult}(\theta, G \setminus z_2, \eta_{(w, w_1)}) = k - 1$ (for z_2 is (θ, w, w_1) -essential in G). So z_2 is (θ, w, w_1) -essential in G' .

Recall that we assume v is (θ, w, w_1) -positive in G . So $\text{mult}(\theta, G \setminus v, \eta_{(w, w_1)}) = k + 1$ and by Corollary 4.6, $\text{mult}(\theta, G \setminus vuz_2, \eta_{(w, w_1)}) \geq k$. From $\eta_{(w, w_1)}(G \setminus v, x) = \eta_{(w, w_1)}(G' \setminus v, x) - |w(e_{uz_2})|^2 \eta_{(w, w_1)}(G \setminus vuz_2, x)$ (part (b) of Theorem 2.2), we deduce that $\text{mult}(\theta, G' \setminus v, \eta_{(w, w_1)}) \geq k$. This means that v is either (θ, w, w_1) -neutral or (θ, w, w_1) -positive in G' .

Now, by part (a) of Lemma 5.2 or part (a) of Lemma 5.3, z_2 is (θ, w, w_1) -essential in $G' \setminus v$. Note that $G' \setminus v$ is a union of two graphs, say G_1 and G_2 , where $u, z_1 \in G_1$ and $z_2 \in G_2$ (for $p = z_1uz_2$ is the only path with endpoints z_1 and z_2 in G). By Lemma 5.9, z_2 is (θ, w, w_1) -essential in $G' \setminus vu$. So $\text{mult}(\theta, G' \setminus vuz_2, \eta_{(w, w_1)}) = k - 1$. But this contradicts the fact that $\text{mult}(\theta, G' \setminus vuz_2, \eta_{(w, w_1)}) = \text{mult}(\theta, G \setminus vuz_2, \eta_{(w, w_1)}) \geq k$ obtained in the preceding paragraph.

Hence v is (θ, w, w_1) -essential in G .

The converse of the lemma follows from Lemma 5.1 and part (a) of Lemma 5.2. \square

Theorem 5.11. *Let u be (θ, w, w_1) -special in G . Then v is (θ, w, w_1) -essential in $G \setminus u$ if and only if v is (θ, w, w_1) -essential in G .*

Proof. For any vertex z , we shall denote its degree by $\deg(z)$. For any graph G_1 , let

$$\chi(G_1) = \sum_{\substack{z \in V(G_1), \\ z \text{ is } (\theta, w, w_1)\text{-special in } G_1}} \deg(z),$$

and m_{G_1} be the number of (θ, w, w_1) -special vertex in G_1 . By Lemma 5.8, $\chi(G_1) \geq 2m_{G_1}$.

We shall prove the theorem by induction on $\chi(G)$. If $\chi(G) = 2m_G$, then by Lemma 5.10, we are done. Assume that the theorem holds for any graph G_1 with $\chi(G_1) < \chi(G)$.

Let $\text{mult}(\theta, G, \eta_{(w, w_1)}) = k$. If $\deg(u) = 2$ in G , we are done by Lemma 5.10. So we may assume $\deg(u) \geq 3$. Let z_1 be a (θ, w, w_1) -essential vertex adjacent to u . Suppose there is a vertex z_3 adjacent to u for which the path $p = z_1uz_3$ is not (θ, w, w_1) -essential in G . Then by Lemma 5.5, u is (θ, w, w_1) -special in G' and $\text{mult}(\theta, G', \eta_{(w, w_1)}) = k$ where $G' = G - e_{uz_3}$.

Suppose for all vertices z' adjacent to u , the path $p' = z_1uz'$ is (θ, w, w_1) -essential in G . Let z_2 and z_4 be adjacent to u . By Lemma 5.5, u is (θ, w, w_1) -positive in G'' and $\text{mult}(\theta, G'', \eta_{(w, w_1)}) = k$, where $G'' = G - e_{uz_4}$. Now $\text{mult}(\theta, G'' \setminus z_1uz_2, \eta_{(w, w_1)}) = k - 1 = \text{mult}(\theta, G \setminus z_1uz_2, \eta_{(w, w_1)})$ (for the path $q = z_1uz_2$ is (θ, w, w_1) -essential in G). So $q = z_1uz_2$ is (θ, w, w_1) -essential in G'' and by part (b) of Lemma 4.9, z_1 and z_2 are (θ, w, w_1) -essential in G'' . This implies that u is (θ, w, w_1) -special in G'' .

Note that in either cases there is a vertex z adjacent to u such that u is (θ, w, w_1) -special in $G''' = G - e_{uz}$, $\text{mult}(\theta, G''', \eta_{(w, w_1)}) = k$ and $\chi(G''') < \chi(G)$.

Now let v be (θ, w, w_1) -essential in $G \setminus u$. Then v is (θ, w, w_1) -essential in $G''' \setminus u = G \setminus u$. By induction hypothesis, v is (θ, w, w_1) -essential in G''' . Therefore $\text{mult}(\theta, G''' \setminus v, \eta_{(w, w_1)}) = k - 1$. By Lemma 5.6, v is either (θ, w, w_1) -positive or (θ, w, w_1) -essential in G . Suppose v is (θ, w, w_1) -positive in G . Then by Corollary 4.6, $\text{mult}(\theta, G \setminus vuz, \eta_{(w, w_1)}) \geq k$. But from $\eta_{(w, w_1)}(G \setminus v, x) = \eta_{(w, w_1)}(G''' \setminus v, x) - |w(e_{uz})|^2 \eta_{(w, w_1)}(G \setminus vuz, x)$ (part (b) of Theorem 2.2), we deduce that $\text{mult}(\theta, G \setminus v, \eta_{(w, w_1)}) = k - 1$, a contradiction. Hence v is (θ, w, w_1) -essential in G .

The converse of the theorem follows from Lemma 5.1 and part (a) of Lemma 5.2. \square

The following Corollary follows from Theorem 5.11 and Lemma 5.2.

Corollary 5.12. (Stability Lemma) *Let G be a graph. If $u \in A_{(\theta, w, w_1)}(G)$ then*

- (i) $D_{(\theta, w, w_1)}(G \setminus u) = D_{(\theta, w, w_1)}(G)$,
- (ii) $P_{(\theta, w, w_1)}(G \setminus u) = P_{(\theta, w, w_1)}(G)$,
- (iii) $N_{(\theta, w, w_1)}(G \setminus u) = N_{(\theta, w, w_1)}(G)$,
- (iv) $A_{(\theta, w, w_1)}(G \setminus u) = A_{(\theta, w, w_1)}(G) \setminus \{u\}$. \square

The next corollary is a generalization of [3, Theorem 1.7] and it can be proved using similar argument as in [3]. Nevertheless we shall give the details of the proof.

Corollary 5.13. (Gallai's Lemma) *Let G be a connected graph for which all vertices are (θ, w, w_1) -essential. Then G is (θ, w, w_1) -critical.*

Proof. Suppose G is not (θ, w, w_1) -critical. Then $\text{mult}(\theta, G, \eta_{(w, w_1)}) = k \geq 2$ (Definition 4.3). Now let $v \in V(G)$. Then $\text{mult}(\theta, G \setminus v, \eta_{(w, w_1)}) = k - 1 \geq 1$. Since G is connected, v is not an isolated vertex. Let u be a vertex adjacent to v in G . By Corollary 4.6, $\text{mult}(\theta, G \setminus uv, \eta_{(w, w_1)}) \geq k - 1$. This implies that u is either (θ, w, w_1) -neutral or (θ, w, w_1) -positive in $G \setminus v$. This also means that all the vertices that are adjacent to v must be either (θ, w, w_1) -neutral or (θ, w, w_1) -positive in $G \setminus v$.

Since $\text{mult}(\theta, G \setminus v, \eta_{(w, w_1)}) \geq 1$, by Lemma 4.4, $G \setminus v$ has at least one (θ, w, w_1) -essential vertex. Together with the conclusion of the previous paragraph, we deduce that $G \setminus v$ has at least one (θ, w, w_1) -special vertex.

Let $A = A_{(\theta, w, w_1)}(G \setminus v)$. By Corollary 5.12, a (θ, w, w_1) -essential vertex remains (θ, w, w_1) -essential, a (θ, w, w_1) -positive vertex remains (θ, w, w_1) -positive and a (θ, w, w_1) -neutral vertex remains (θ, w, w_1) -neutral, upon deletion of a (θ, w, w_1) -special vertex. Also a (θ, w, w_1) -special vertex remains (θ, w, w_1) -special, upon deletion of a (θ, w, w_1) -special vertex. Therefore if H is a component in $(G \setminus v) \setminus A$, either $\text{mult}(\theta, H) > 0$ and all the vertices in H are (θ, w, w_1) -essential, or $\text{mult}(\theta, H) = 0$.

Let $Q_1, Q_2, \dots, Q_l, T_1, T_2, \dots, T_m$ be all the components in $(G \setminus v) \setminus A$ where $\text{mult}(\theta, Q_j) > 0$ and $\text{mult}(\theta, T_{j'}) = 0$ for all j, j' . By part (a) of Theorem 2.2, we deduce that

$$\text{mult}(\theta, (G \setminus v) \setminus A, \eta_{(w, w_1)}) = \sum_{j=1}^l \text{mult}(\theta, Q_j, \eta_{(w, w_1)}).$$

On the other hand, by applying Corollary 5.12 repeatedly (also Lemma 5.1), $\text{mult}(\theta, (G \setminus v) \setminus A) = k - 1 + |A|$. So $\sum_{j=1}^l \text{mult}(\theta, Q_j) = k - 1 + |A|$.

Let $a \in A$. Note that v is not adjacent to any vertices in $\bigcup_i Q_i$ (by the conclusion of the first paragraph). Therefore all the Q_i 's are components in $G \setminus A$. Since $\text{mult}(\theta, G \setminus a, \eta_{(w, w_1)}) = k - 1$, by applying Lemma 4.2 repeatedly,

$$\text{mult}(\theta, (G \setminus a) \setminus (A \setminus \{a\}), \eta_{(w, w_1)}) \leq k - 1 + |A \setminus \{a\}| = k - 2 + |A|.$$

Again by part (a) of Theorem 2.2, we deduce that $\sum_{j=1}^l \text{mult}(\theta, Q_j, \eta_{(w, w_1)}) \leq k - 2 + |A|$, a contradiction. Hence $k = 1$ and G is (θ, w, w_1) -critical. \square

As a consequence of Corollary 5.12 and Corollary 5.13, we have the following;

Corollary 5.14. *Let $A = A_{(\theta, w, w_1)}(G)$. Then*

- (a) $A_{(\theta, w, w_1)}(G \setminus A) = \emptyset$, $D_{(\theta, w, w_1)}(G \setminus A) = D_{(\theta, w, w_1)}(G)$, $P_{(\theta, w, w_1)}(G \setminus A) = P_{(\theta, w, w_1)}(G)$, and $N_{(\theta, w, w_1)}(G \setminus A) = N_{(\theta, w, w_1)}(G)$.
- (b) $G \setminus A$ has exactly $(|A| + \text{mult}(\theta, G, \eta_{(w, w_1)}))$ (θ, w, w_1) -critical components.
- (c) If H is a component of $G \setminus A$ then either H is (θ, w, w_1) -critical or $\text{mult}(\theta, H, \eta_{(w, w_1)}) = 0$.
- (d) The subgraph induced by $D_{(\theta, w, w_1)}(G)$ consists of all the (θ, w, w_1) -critical components in $G \setminus A$.

\square

6 Connection with classical Gallai-Edmonds decomposition

Definition 6.1. The *deficiency* of a graph, denoted by $\text{def}(G)$ is the number of vertices left uncovered by any maximum matching in G .

The following lemma follows easily from Lemma 1.1.

Lemma 6.2. For any edge weight function w , $\text{mult}(0, G, \mu_w) = \text{def}(G)$. □

By Lemma 6.2, the edge weight function has no effect on the multiplicity of 0 as a root in $\mu_w(G, x)$. Assume that $w_1(u) = 0$ for all $u \in V(G)$. Then by Lemma 1.6, $\eta_{(w, w_1)}(G, x) = \mu_w(G, x)$. Note also that $D_{(0, w, w_1)}(G)$ is the set of all vertices in G which are not covered by at least one maximum matching of G . Also $N_{(0, w, w_1)}(G) = \emptyset$, for otherwise, there would be a vertex say u with $\text{mult}(0, G, \mu_w) = \text{mult}(0, G \setminus u, \mu_w)$. But this means that there is a maximum matching that does not cover u and so $u \in D_{(0, w, w_1)}(G)$, a contradiction (see [22, Section 3.2 on p. 93] for the details). Therefore $V(G) = D_{(0, w, w_1)}(G) \cup A_{(0, w, w_1)}(G) \cup P_{(0, w, w_1)}(G)$ which is the classical Gallai-Edmonds decomposition provided $w(e) = 1$ for all $e \in E(G)$. Also [22, Lemma 3.2.2 on p. 95] is a special case of the Stability Lemma (Corollary 5.12).

Lemma 6.3. Suppose $w_1(u) = c$ for all $u \in V(G)$, where c is a constant real number. Then $\eta_{(w, w_1)}(G, x + c) = \mu_w(G, x)$.

Proof. We shall prove by induction on $|V(G)|$. Suppose $|V(G)| = 1$. Then $\eta_{(w, w_1)}(G, x) = x - c$. Therefore $\eta_{(w, w_1)}(G, x + c) = x = \mu_w(G, x)$. Assume it is true for all graphs with fewer vertices than G .

Let $u, v \in V(G)$. By induction, $\eta_{(w, w_1)}(G \setminus u, x + c) = \mu_w(G \setminus u, x)$ and $\eta_{(w, w_1)}(G \setminus uv, x + c) = \mu_w(G \setminus uv, x)$. So $\eta_{(w, w_1)}(G, x + c) = x\mu_w(G \setminus u, x) - \sum_{v \sim u} |w(e_{uv})|^2 \mu_w(G \setminus uv, x)$ (by part (c) of Theorem 2.2). It then follows from by part (c) of Lemma 2.1, that $\eta_{(w, w_1)}(G, x + c) = \mu_w(G, x)$. □

The next lemma follows from Lemma 6.2 and Lemma 6.3.

Lemma 6.4. Suppose $w_1(u) = c$ for all $u \in V(G)$, where c is a constant real number. Then for any edge weight function w , $\text{mult}(c, G, \eta_{(w, w_1)}) = \text{def}(G)$. □

As a consequence of Lemma 6.4, we see that if the weight on each vertex is a constant, say c , then the edge weight function has no effect on the multiplicity of c as a root of $\eta_{(w, w_1)}(G, x)$. In fact, it depends only on the structure of the graph. Note also that $D_{(c, w, w_1)}(G)$ is the set of all vertices in G which are not covered by at least one maximum matching of G , $N_{(c, w, w_1)}(G) = \emptyset$ and $V(G) = D_{(c, w, w_1)}(G) \cup A_{(c, w, w_1)}(G) \cup P_{(c, w, w_1)}(G)$.

7 Connection with the Parter-Wiener theorem

In separate papers, Parter [24] and Wiener [25] independently observed an important theorem about the existence of principal submatrices of a Hermitian matrix whose graph is a tree, in which the multiplicity of an eigenvalue increases. Recently, Johnson, Duarte and Saiago [11] generalized this

result by providing more structural information. It turns out that these results are just special cases of the Gallai-Edmonds structure theorem that we have developed in this paper.

Given an $n \times n$ Hermitian matrix $B = [b_{uv}]$, we can associate a graph G to it as follows: let G be the graph with $V(G) = \{1, 2, \dots, n\}$ and $e_{uv} \in E(G)$ if and only if $b_{uv} \neq 0$, $u \neq v$. Clearly, for a given graph G , there are many Hermitian matrix $B = [b_{uv}]$ whose associated graph is G ; moreover, we shall assign weights to G using B as follows: set $w(e_{uv}) = b_{uv}$ if $e_{uv} \in E(G)$ with $u < v$ and set $w_1(u) = b_{uu}$ for all $u \in V(G)$. Consequently, $B = B_{(w, w_1)}(G)$ is the weighted adjacency matrix of G so that the characteristic polynomial of B is $\phi_{(w, w_1)}(G, x)$ and the eigenvalues of B are the roots of $\phi_{(w, w_1)}(G, x)$.

For the rest of this section, let T be a tree on n vertices $1, 2, \dots, n$ and suppose that $\mathcal{S}(T)$ is the set of all $n \times n$ Hermitian matrix whose graph is T . Let $m_B(\lambda)$ denote the multiplicity of λ as an eigenvalue of B . Suppose $B \in \mathcal{S}(T)$. Let $B(u; u)$ be the matrix obtained from B by deleting the u -th row and u -column. Note that $B(u; u) = B_{(w, w_1)}(T \setminus u)$.

In the literature, a vertex v of a tree T is called *parter* if $m_{B(v; v)}(\lambda) = m_B(\lambda) + 1$, *neutral* if $m_{B(v; v)}(\lambda) = m_B(\lambda)$ and *downer* if $m_{B(v; v)}(\lambda) = m_B(\lambda) - 1$ (see [10, 11, 12, 13]). On the other hand, by Corollary 2.12, $\phi_{(w, w_1)}(T, x) = \eta_{(w, w_1)}(T, x)$. So parter, neutral and downer (relative to the eigenvalue λ) are just (λ, w, w_1) -positive, (λ, w, w_1) -neutral and (λ, w, w_1) -essential respectively, in the language of Godsil (Definition 4.3). Furthermore, $m_B(\lambda) = \text{mult}(\lambda, T, \eta_{(w, w_1)})$.

The following result has been important in the recent development on possible multiplicities of eigenvalues among matrices in $\mathcal{S}(T)$ [10, 11, 12, 13].

Theorem 7.1 (Parter-Wiener). *Let T be a tree on n vertices and suppose $B \in \mathcal{S}(T)$ and $\lambda \in \mathbb{R}$ is such that $m_B(\lambda) \geq 2$. Then, there is a vertex u of T such that $m_{B(u; u)}(\lambda) = m_B(\lambda) + 1$ and λ occurs as an eigenvalue in direct summands of B that correspond to at least three branches of T at u . \square*

A more general statement was proved by Johnson, Duarte and Saiago [11] as follows.

Theorem 7.2 (Johnson-Duarte-Saiago). *Let B be a Hermitian matrix whose graph is T , and suppose that there exists a vertex u of T and a real number λ such that $\lambda \in \sigma(B) \cap \sigma(B(u; u))$ (here $\sigma(B)$ denotes the set of all eigenvalues of B). Then*

- (a) *there is a vertex v of T such that $m_{B(v; v)}(\lambda) = m_B(\lambda) + 1$;*
- (b) *if $m_B(\lambda) \geq 2$, then v may be chosen so that the degree of v is at least 3 and so that there are at least three components T_1 , T_2 and T_3 of $T - v$ such that $m_{B[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$ (here $B[T_i]$ is the principal submatrix of B from retention of the rows and columns which correspond to T_i);*
- (c) *if $m_B(\lambda) = 1$, then v may be chosen so that the degree of v is at least 2 and so that there are two components T_1 and T_2 of $T - v$ such that $m_{B[T_i]}(\lambda) = 1$, $i = 1, 2$.*

Proof. We sketch a proof based on the Gallai-Edmonds structure theorem (Corollary 5.12, Corollary 5.13, Corollary 5.14). Note that the condition $\lambda \in \sigma(B) \cap \sigma(B(u; u))$ means that either $m_B(\lambda) \geq 2$ or $m_B(\lambda) = 1$ and u is not (λ, w, w_1) -essential in T . In the later, we observe that $A_{(\lambda, w, w_1)}(T) \neq \emptyset$ since T is connected. In fact, this is equivalent to saying that λ is a root of $\eta_{(w, w_1)}(T, x)$ and $A_{(\lambda, w, w_1)}(T) \neq \emptyset$ (see Corollary 5.13). We shall consider these cases separately.

Case I. $m_B(\lambda) \geq 2$.

If $A_{(\lambda,w,w_1)}(T) = \{v\}$ then it follows immediately from part (b) of Corollary 5.14 that v has the property required by (b). We proceed to prove the theorem by induction on $|A_{(\lambda,w,w_1)}(T)|$.

Fix $v \in A_{(\lambda,w,w_1)}(T)$ and consider a component T' of $T \setminus v$ with $A_{(\lambda,w,w_1)}(T') \neq \emptyset$. Such T' exists since $A_{(\lambda,w,w_1)}(T \setminus v) = A_{(\lambda,w,w_1)}(T) \setminus v \neq \emptyset$ (Corollary 5.12). By the inductive hypothesis, there exists $v' \in A_{(\lambda,w,w_1)}(T')$ such that $T' \setminus v'$ consists of $T'_1, \dots, T'_k, S'_1, \dots, S'_l$ with $m_{B[T'_i]}(\lambda) \geq 1$ for all $1 \leq i \leq k$, $k \geq 3$, and $m_{B[S'_i]}(\lambda) = 0$ for all $1 \leq i \leq l$ (it is possible that there does not exist any S'_i).

Since T is a tree, v is joined to at most one of T'_i 's, S'_j 's or v' . If v is not joined to any of the T'_i 's then we are done; otherwise, we may assume that $k = 3$ and v is joined to T'_3 . Set $T_1 = T'_1$, $T_2 = T'_2$ and T_3 to be the component of $T \setminus v'$ containing T'_3 and v . It is readily deduced from $m_{B[T \setminus v']}(\lambda) \geq 3$ that $m_{B[T_i]}(\lambda) \geq 1$ for $i = 1, 2, 3$, as desired.

Case II. $m_B(\lambda) = 1$.

Since a similar argument can be used to settle this case, we omit the details. □

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